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CLASSICAL MECHANICS

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PHI

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To

Myrtle and our children

Vinod & Anitha, Manoj & Bini, Ann & Suresh

Contents

Preface

Man has tried, from time immemorial, to increase his understanding of the world in which he lives. A crowning achievement in this attempt was the creation of Classical Mechanics by Newton, Lagrange, Hamilton and others. Classical Mechanics is based on the concept that each system has a definite position and momentum. Mechanics is usually the first course, following introductory physics, at the degree level for students of physics, mathematics, and engineering. A thorough understanding of mechanics serves as a foundation for studying different areas in these branches. The study of Classical Mechanics also gives the students an opportunity to master many of the mathematical techniques.

The book is an outgrowth of the lectures on Classical Mechanics which the author had given for a number of years at the postgraduate level in different universities in Kerala, and as such the material is thoroughly class-tested. It is designed as a textbook for one-semester courses for postgraduate students of physics, mathematics and engineering. I have made every effort to organize the material in such a way that abstraction of the theory is minimized. Details of mathematical steps are provided wherever found necessary. Every effort has been taken to make the book explanatory, exhaustive and user-friendly.

In the conventional approach to the subject, Lagrangian and Hamiltonian formulations are usually taught at the end of the course. However, I have introduced these topics at an early stage, so that the students become familiar with these methods. Chapters 1 and 2 are of introductory nature, discussing mainly the different frames of reference and the Newtonian mechanics of a single particle and system of particles. In the next two chapters, Lagrange's formalism and the variational principle have been presented with special emphasis on generalized coordinates, Lagrange's equation, first integrals of motion, Lagrange multiplier method, and velocity-dependent potentials, which are needed for the study of electromagnetic force. A section on symmetry properties and conservation laws, which leads to the important pairs of dynamical variables that follow the uncertainty principle in quantum mechanics, is also presented. Chapter 5 on central force motion has been broadened to include topics like satellite parameters, communication satellites, orbital transfers, and scattering problem.

Hamilton's formulation of mechanics along with Hamilton–Jacobi method, Chapters 6 and 7, provides a framework to discuss the dynamics of systems in the phase space. The technique of action-angle variables leads to Wilson–Sommerfeld quantum condition, which is an essential rule in quantum theory. Poisson bracket, an integral part of classical mechanics, is also indispensable for the formulation of quantum mechanics. Rigid body motion, Euler's angles, Coriolis force, Euler's equations of motion, and motion of symmetric tops have all been discussed in Chapter 8. A chapter on the essentials of small oscillations, which is crucial for the study of molecular vibrations, is also included. Further, a chapter on special theory of relativity is presented to enable the study of systems moving at relatively high velocities. This chapter discusses Lorentz transformation, relativistic dynamics, space–time diagram, four-vectors, and invariance of Maxwell's equations.

To provide a smooth transition from the traditional topics of Classical Mechanics to the modern ones, two chapters (11 and 12) on the rapidly growing areas of nonlinear dynamics and chaos have also been included in the book.

Learning to solve problems is the basic purpose of any course, since this process helps in understanding the subject. Keeping this in mind, considerable attention is devoted to worked examples illustrating the concepts involved. Another notable feature of the book is the inclusion of end-of-chapter review questions and problems. These provide the instructor with enough material for home assignment and classroom discussions. Answers to all problems are given at the end of the book. The usual convention of indicating vectors by boldface letters is followed. A solutions manual is available from the publisher for the use of teachers.

The saying 'I have learnt much from my teachers but more from my pupils' rings true in the context of writing this book. I wish to record my gratitude to my students for their active participation in the discussions we had on various aspects of the subject. I place on record my gratitude to Dr. V.K.Vaidyan, Dr. V.U. Nayar, Dr. C.S. Menon, Dr. V. Ramakrishnan, Dr. V.S. Jayakumar, Lisha R. Chandran and Simitha Thomas for their interest and encouragement during the preparation of the book. Finally, I express my sincere thanks to the publisher, PHI Learning, for their unfailing cooperation and for the meticulous processing of the manuscript.

Above all I thank my Lord Jesus Christ, who has given me wisdom, might and

guidance all through my life.

G. Aruldhas

1

Introduction to

Newtonian Mechanics Classical mechanics deals with the motion of physical bodies at the macroscopic level. Galileo and Sir Isaac Newton laid its foundation in the 17th century. As Newton's laws of motion provide the basis of classical mechanics, it is often referred to as *Newtonian mechanics*. **There are two parts in mechanics:** *kinematics***and** *dynamics*. **Kinematics deals with the geometrical description of the motionof objects without considering the forces producing the motion. Dynamics is the part that concerns**

the forces that produce changes in motion or the changes in other properties. This leads us to the concept of force, mass and the laws that govern the motion of objects. To apply the laws to different situations, Newtonian mechanics has since been reformulated in a few different forms, such as the Lagrange, the Hamilton and the Hamilton-Jacobi formalisms. All these formalisms are equivalent and their applications to topics of interest form the basis of this book.

1.1 FRAMES OF REFERENCE

The most basic concepts for the study of motion are space and time, both of which are assumed to be continuous. To describe the motion of a body, one has to specify its position in space as a function of time. To do this, a co-ordinate system is used as a frame of reference. One convenient co-ordinate system we frequently use is the cartesian or rectangular co-ordinate system.

Cartesian Co-ordinates (x, y, z) The position of a point P in a

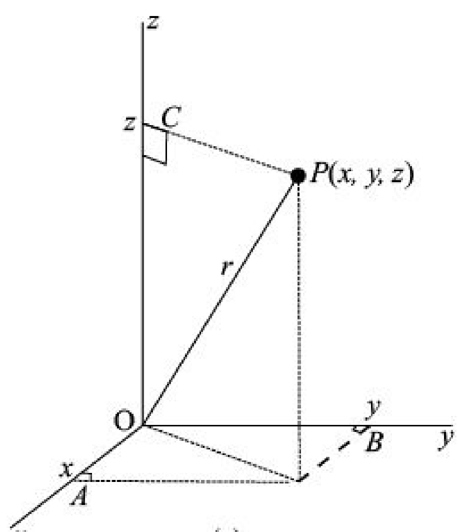
cartesian co-ordinate system, as shown in Fig. 1.1(a), is specified by three co-ordinates (x, y, z) or (x_1, x_2, x_3) or by the position vector r. A vector quantity will be denoted by boldface type (as r), while the magnitude will be represented by the symbol itself (as r). A unit vector in the direction of the vector r is denoted by the corresponding letter with a circumflex over it (as $\hat{r} = \mathbf{r}/r$). In terms of the co-ordinates, the vector and the magnitude of the vector are given by

$$\mathbf{r} = \hat{i}x + \hat{j}y\ \hat{k}z$$

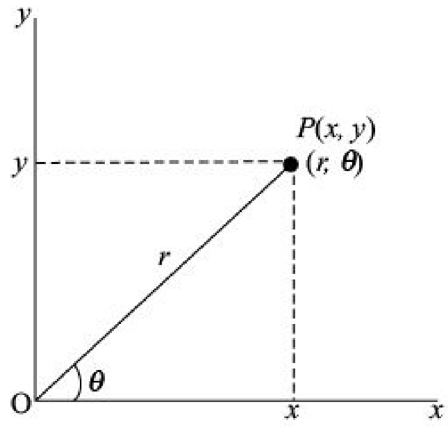
$$r^2 = x^2 + y^2 + z^2 \tag{1.1}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the rectangular axes x, y and z respectively.

Elementary lengths in the direction of x, y, z: dx, dy, dz Elementary volume: $dx\ dy\ dx$ Cartesian co-ordinates are convenient in describing the motion of objects in a straight line. However, in certain problems, it is convenient to use non-rectangular co-ordinates.



(a)



(b)

Fig. 1.1 (a) Cartesian co-ordinates (*x*, *y*, *z*) of a point *P* in three dimensions; (b) Plane polar co-ordinates (*r*, *q*) of a point *P*.

Plane Polar Co-ordinates (*r*, *q*) To study the motion of a particle in a plane, the plane co-ordinate system which is shown in Fig. 1.1 (b) is probably the proper choice. The radius vector of the point *P* in the *xy* plane is r. The line *OP* makes an angle *q* wih the x-axis. The position of point *P* can be described by the co-ordinates (*r*, *q*) called plane polar co-ordinates. The rectangular co-ordinates of *P* are (*x*, *y*). The relations connecting (*x*, *y*) and (*r*, *q*) can be written from Fig. 1.1 (b) as:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{1.2}$$

From Eq. (1.2)

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.3}$$

Elementary lengths in the direction of increasing *r* and *q*: *dr*, *rdq*

Elementary area: $dr \times rd\theta = r\,dr\,d\theta$

Velocity components: $\dot{r}$ and $r\dot{\theta}$

Thus, in a two dimensional system (x, y) or (r, θ) completely specifies the position of a point in a plane.

Cylindrical Co-ordinates (*r*, *f*, z) Consider a point *P* having a radius vector r. Point *P* can be specified by using a set of cartesian

co-ordinates (*x*, *y*, *z*) or cylindrical co-ordinates (*r*, *f*, *z*) as shown in Fig. 1.2 (a). The co-ordinate *r* is the projection of the radius vector r on the *xy*-plane. The two sets of co-ordinates are related by the relations:

$$x = \rho \cos \phi \qquad \rho^2 = x^2 + y^2$$

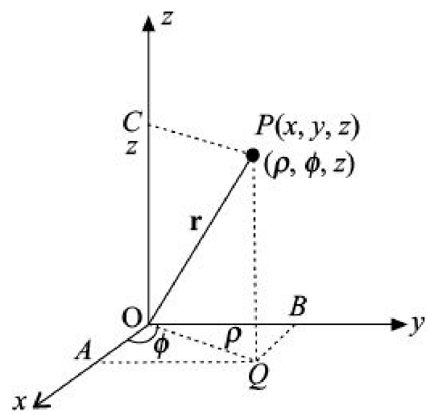
$$y = \rho \sin \phi \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.4}$$

Elementary lengths: $d\rho$, $\rho d\phi$, dz

Elementary volume: $\rho\, d\rho\, d\phi\, dz$

Velocity components: $\dot{\rho}$, $\rho\dot{\phi}$, $\dot{z}$

Spherical Polar Co-ordinates (*r*, *q*, *f*) Figure 1.2 (b) defines the spherical polar co-ordinates of a point *P* having a radius vector r. The cartesian co-ordinates of *P* are (*x*, *y*, *z*). The co-ordinate *q* is the angle that the radius vector *r* makes with the *z*-axis and *f* is the angle that the projection of the position vector into the *xy*-plane makes with the *x*-axis. From Fig. 1.2 (b).



(a)

OQ = *r* sin *q* and *OC* = *PQ* = *r* cos *q*

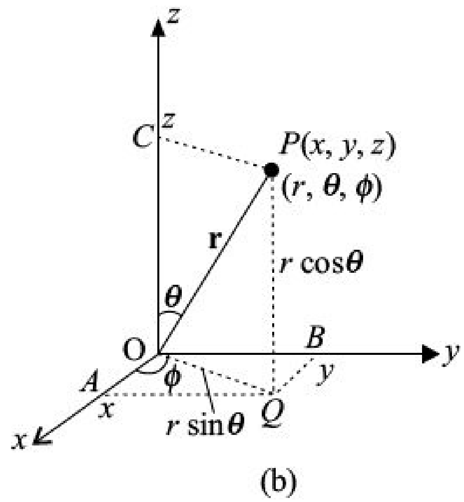


Fig. 1.2 (a) Cylindrical co-ordinates (*r*, *f*, *z*) of a point *P* in space: (b) Spherical polar co-ordinates (*r*, *q*, *f*) of a point *P* in space.

The two sets of co-ordinates are related by the relations:

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta \tag{1.5}$$

Elementary lengths: dr, $rd\theta$, $r \sin\theta\, d\phi$

Elementary volume: $dr \times rd\theta \times r \sin\theta\, d\phi = r^2 \sin\theta\, d\theta\, d\phi\, dr$

Velocity components: $\dot{\mathbf{r}}$, $r\dot{\theta}$, $r \sin\theta\, \dot{\phi}$

1.2 NEWTON'S LAWS OF MOTION

Newton's First Law of Motion *Every object continues in its state of rest or uniform motion in a straight line unless a net external force acts on it to change that state.*

Newton's first law indicates that the state of a body at rest (zero velocity) and a state of uniform velocity are completely equivalent. No external force is needed in order to maintain the uniform motion of a body; it continues without change due to an intrinsic property of the body that we call *inertia*. Because of this property, the first law is often referred to as the *law of inertia*. **Inertia** is the natural tendency of a body to remain at rest or in uniform motion along a straight line. Quantitatively, the inertia of a body is measured by its **mass**. In one sense, Newton made the first law more precise by introducing definitions of quantity of motion and amount of matter which we now call *momentum* and *mass* respectively. The momentum of a body is simply proportional to its velocity. The coefficient of proportionality is a constant for any given body and is called its mass. Denoting mass by m and momentum vector by $\mathbf{p}$ $\mathbf{p} = m\mathbf{v}$ (1.6) where $\mathbf{v}$ is the velocity of the body. Mathematically, Newton's first law can be expressed in the following way. In the absence of an external force acting on a body $\mathbf{p} = m\mathbf{v} =$ constant (1.7) This is the **law of conservation of momentum**. As per the special theory of relativity (see Section 10.10), mass is not a constant but varies with velocity.

Newton's Second Law of Motion *The rate of change of momentum of an object is directly proportional to the force applied and takes place in the direction of the force.*

If we denote the force by $\mathbf{F}$, then the second law can be written mathematically as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{1.8}$$

Since $\mathbf{p} = m\mathbf{v}$ and m is a constant, the second law can also be written as

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{1.9}$$

where $\mathbf{a}$ stands for acceleration. Since $\mathbf{v} = d\mathbf{r}(t)/dt$, the second law may also be written as

$$\mathbf{F} = m\frac{d^2\mathbf{r}(t)}{dt^2} \tag{1.10}$$

which is often referred to as the **equation of motion of the particle.** It is a second order differential equation. If the force $\mathbf{F}$ is known and the position and velocity of the particle at a particular instant are given, with the help of second law we can find the position and velocity of the particle at any given instant. That is, its path is completely known if accurate values of its co-ordinates and velocity (or momentum $\mathbf{p} = m\mathbf{v}$) at a particular instant are known simultaneously. In quantum mechanics, we will be learning that this deterministic model is not applicable to atomic and subatomic particles.

Newton's Third Law of Motion *Whenever a body exerts a force on a second body, the second exerts an equal and opposite force on the* *first.*

This law is often paraphrased as *to every action there is an equal and opposite reaction*. This statement is perfectly valid but it has to be remembered that the action force and the reaction force are acting on different bodies. In a twoparticle system, the force acting on particle 1 by particle 2, $\mathbf{F}_{12}$, is equal and opposite to the force acting on particle 2 by particle 1, $\mathbf{F}_{21}$. That is, $\mathbf{F}_{12} = -\mathbf{F}_{21}$

Since force is the rate of change of momentum

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt}$$

or
$$m_1\mathbf{a}_1 = -m_2\mathbf{a}_2 \tag{1.11}$$

or
$$\frac{m_2}{m_1} = \left|\frac{\mathbf{a}_1}{\mathbf{a}_2}\right| \tag{1.12}$$

Equation (1.12) can be used to determine the mass of particles.

1.3 INERTIAL AND NON-INERTIAL FRAMES

Newton's first law does not hold in every reference frame. When two bodies fall side by side, each of them is at rest with respect to the other while at the same time it is subject to the force of gravity. Such cases would contradict the stated first law. Reference frames in which Newton's law of inertia holds good are called **inertial reference frames**. The remaining laws are also valid in inertial reference frames only. The acceleration of an inertial reference frame is zero and therefore it moves with a constant velocity. Any reference frame that moves with constant velocity relative to an inertial frame is also an inertial frame of reference. For simple applications in the laboratory, reference frames fixed on the earth are inertial frames. For astronomical applications, the terrestrial frame cannot be regarded as an inertial frame. A reference frame where the law of inertia does not hold is called a **non-inertial reference frame**.

The accelerations in Eq. (1.12) can be measured experimentally. Hence, Eq. (1.12) can be used to determine the mass of a particle by selecting *m*1 as unit mass. The mass of a body determined in this way is termed as its **inertial mass** because it characterizes the inertial properties of bodies. Mass can also be defined on the basis of Newton's law of gravitation. The mass of a body defined on the basis of gravitational properties is called the **gravitational mass**. Naturally a question arises: Is the inertial mass of a body equal to its gravitational mass? Recently it was established that these masses are equal to within a few parts in 10^{12}. This equivalence of the inertial and gravitational masses of a body is the **principle of equivalence** postulated by Einstein in general relativity.

1.4 MECHANICS OF A PARTICLE

In this section, we shall discuss mainly the conservation laws for a particle in motion in Newtonian formalism.

Conservation of Linear Momentum From Newton's first law, we have already indicated the law of conservation of momentum of a single particle in Eq. (1.7). It also follows from Newton's second law of motion which states that

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$

If no external force is acting on the particle

$$\frac{d\mathbf{p}}{dt} = 0 \quad \text{or} \quad \mathbf{p} = \text{constant in time} \tag{1.13}$$

If the total force acting on a particle is zero, then the linear momentum **p** *is conserved.*

Angular Momentum and Torque Angular momentum and torque are two important quantities in rotational motion. A force causes linear acceleration whereas a torque causes angular acceleration. The *angular momentum* **of a particle about a point O (say origin), denoted by**
L, is defined as L = r p (1.14) where r is the radius vector of the particle. The *torque* **(N) or moment of a force about O is**

defined as $$\mathbf{N} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \quad (1.15)$$

which is perpendicular to the plane containing the vectors **r** and **F** points in the direction of the advance of a right hand screw from **r** to **F**. Since

$$\frac{d\mathbf{r}}{dt} \times \mathbf{p} = m\frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = 0$$

from Eq. (1.15), we have

$$\mathbf{N} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt}$$

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} \quad (1.16)$$

which is the analogue of Newton's second law in rotational motion.

Conservation of Angular Momentum The angular momentum conservation comes automatically from Eq. (1.16).
If the torque N acting on the particle is zero, then

$$\frac{d\mathbf{L}}{dt} = 0 \quad \text{or} \quad \mathbf{L} = \text{constant} \quad (1.17)$$

If the torque **N** *acting on a particle is zero, the angular momentum* **L** *is a constant.* Planets moving around the sun and satellites around the earth are some of the very common examples.

Work Done by a Force Work done by an external force in moving a particle from position 1 to position 2 is given by

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 m\frac{d\mathbf{v}}{dt} d\mathbf{r}$$

Assuming the mass of the particle constant

$$W_{12} = m\int_1^2 \frac{d\mathbf{v}}{dt}\frac{d\mathbf{r}}{dt} dt = m\int_1^2 \frac{d\mathbf{v}}{dt}\mathbf{v}\, dt$$

$$= m\int_1^2 \mathbf{v}\, d\mathbf{v} = \frac{m}{2}\int_1^2 d(\mathbf{v}^2) = \frac{m}{2}(\mathbf{v}_2^2 - \mathbf{v}_1^2)$$

$$= T_2 - T_1 \tag{1.18}$$

where T_2 and T_1 are the kinetic energies of the particle in positions 2 and 1 respectively. If $T_2 > T_1$, $W_{12} > 0$, work is done by the force on the particle and as a result the kinetic energy of the particle is increased. If $T_1 > T_2$, $W_{12} < 0$, work is done by the particle against the force and as a result the kinetic energy of the particle is decreased.

Conservative Force If the force acting on a system is such that the work done along a closed path is zero, then the force is said to be conservative. That is, for a conservative force F

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \tag{1.19}$$

If the closed curve encloses the surface **S**, by Stokes theorem, we have

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_s (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0 \tag{1.20}$$

Since the surface is arbitrary, this is possible only if

$$\nabla \times \mathbf{F} = Curl\ \mathbf{F} = 0 \tag{1.21}$$

which is the necessary and suficient condition for a force to be conservative. The curl of a vector is zero if it can be expressed as the gradient of a scalar function of position. Hence, we can write

$$\mathbf{F} = -\nabla V(\mathbf{r}) \tag{1.22}$$

The scalar function $V(\mathbf{r})$ in Eq. (1.22) is called the **potential energy** of the particle at the point or simply the **potential** at the point. In terms of V, the components of the force are

$$F_x = \frac{\partial V}{\partial x} \qquad F_y = \frac{\partial V}{\partial y} \qquad F_z = \frac{\partial V}{\partial z} \tag{1.23}$$

Conservation of Energy The work done by a force F in moving a particle of mass *m* from position 1 to position 2 is given by Eq.

(1.18). Now consider the work done W_{12} by taking F to be a conservative force derivable from a potential V. Then W_{12} takes the form $W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = -\int_1^2 \nabla V \cdot d\mathbf{r}$

$$= -\int_1^2 \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = -\int_1^2 dV$$

$$= V_1 - V_2 \qquad (1.24)$$

Combining Eqs. (1.18) and (1.24), we have $T_1 + V_1 = T_2 + V_2$
which gives the energy conservation theorem.

If the force acting on a particle is conservative, then the total energy of the particle, $T + V$, is a constant.

Equation (1.22) is satisfied even if we replace V by $V + C$, where C is a constant. Then $F = -\nabla V = -\nabla(V + C) \qquad (1.25)$

Hence, the potential introduced through Eq. (1.22) is not unique and therefore an absolute value of the potential has no meaning. It may be noted that the kinetic energy also has no absolute value since we use an inertial frame of reference for measuring the velocity and hence the kinetic energy. For measuring the absolute kinetic energy we required a reference frame which is absolutely at rest. It is not possible to find such a reference frame and therefore the kinetic energy we measure is only relative.

1.5 MOTION UNDER A CONSTANT FORCE

When the applied force **F** on a particle is constant in time and hence there is a constant acceleration, we write Eq. (1.10) in the form

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} = \mathbf{a} = \text{constant} \qquad (1.26)$$

Direct integration of Eq. (1.26) is possible if the initial conditions are known.

With the initial condition $\mathbf{v} = \mathbf{v}_0$ at $t = 0$, on integrating Eq. (1.26) we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t \tag{1.27}$$

Since $\mathbf{v} = d\mathbf{r}/dt$, assuming the initial conditions $\mathbf{r} = \mathbf{r}_0$ at $t = 0$ leads to

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2 \tag{1.28}$$

Substituting t from Eq. (1.27) in Eq. (1.28)

$$\mathbf{v}^2 = \mathbf{v}_0^2 + 2\mathbf{a}(\mathbf{r} - \mathbf{r}_0) \tag{1.29}$$

In one dimension, Eqs. (1.27), (1.28) and (1.29) reduce to

$$\mathrm{v}(t) = \mathrm{v}_0 + at \tag{1.30}$$

$$x = x_0 + \mathrm{v}_0 t + \frac{1}{2}at^2 \tag{1.31}$$

$$\mathrm{v}^2 = \mathrm{v}_0^2 + 2a(x - x_0) \tag{1.32}$$

Equations (1.30), (1.31) and (1.32) are the familiar equations that describe the translational motion of a particle in one dimension. One of the familiar exmples of motion under a constant force is motion under gravity.

1.6 MOTION UNDER A TIME-DEPENDENT FORCE

When the force acting on a particle is an explicit function of time, the equation of motion can be written as

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}(t) \tag{1.33}$$

Assuming the initial conditions that $\mathbf{v} = \mathbf{v}_0$ at $t = t_0$, integration gives

$$\mathbf{v}(t) = \mathbf{v}_0 + \frac{1}{m}\int_{t_0}^{t} \mathbf{F}(t)dt \tag{1.34}$$

As $\mathbf{v} = d\mathbf{r}/dt$, integration of Eq. (1.34) gives $\mathbf{r}(t)$, assuming $\mathbf{r} = \mathbf{r}_0$ at $t = t_0$, as

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{m}\int_{t_0}^{t} dt' \int_{t_0}^{t'} \mathbf{F}(t'')dt'' \tag{1.35}$$

Since there is a double integration, two variables *t* and *t* are used. If explicit integration of the integrals in Eq. (1.35) is not possible, one has to go for numerical integration.

1.7 REFLECTION OF RADIOWAVES FROM THE IONOSPHERE

To illustrate the motion under a time-dependent force, we consider the interaction of radiowaves with electrons in the ionosphere. Ionosphere is a region that surrounds the earth at a height of approximately 200 km from the surface of the earth. It consists of positively charged ions and negatively charged electrons which are formed when ultraviolet rays from the sun is absorbed by atoms and molecules of the upper atmosphere. The particles are trapped by the earth's magnetic field and stay in the upper region, forming the **ionosphere** which is electrically neutral. When a radiowave, which is an electromagnetic wave, passes through the ionosphere, it interacts with the charged particles and accelerates them. Since electrons are much lighter than the positive ions, they are more effective in modifying the propagation of the radiowaves.

The electric field **E** of the electromagnetic plane wave is given by $\mathbf{E} = \mathbf{E}_0 \sin wt$ (1.36) where *w* is the angular frequency of the wave. The ionosphere may be regarded as a region made of free electron gas. A free electron of charge – *e* interacts with the electric field **E** which results in a force on the electron: $\mathbf{F} = -e\mathbf{E} = -e\,E_0 \sin wt$ (1.37) The acceleration of the electron is

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{d\mathbf{v}}{dt} = -\frac{e\mathbf{E}_0}{m}\sin\omega t \qquad (1.38)$$

which can be integrated to give the velocity of the electron as a function of time:

$$\mathbf{v}(t) = \mathbf{v}_0 - \frac{e\mathbf{E}_0}{m} + \frac{e\mathbf{E}_0}{m\omega}\cos\omega t \qquad (1.39)$$

where $\mathbf{v}_0$ is the velocity of the electron at $t = t_0$. Since $\mathbf{v} = dx/dt$ assuming that $x = x_0$ at $t = t_0$, integration of Eq. (1.39) gives the position of the electron as a function of time:

$$x(t) = x_0 + \left(\mathrm{v}_0 - \frac{eE_0}{m\omega}\right)t + \frac{eE_0}{m\omega}\sin\omega t \qquad (1.40)$$

The first two terms indicate that the electron is drifting with a uniform velocity which is a function of the initial conditions only. Superimposed on this drifting motion is an oscillating motion represented by the last term. The oscillating frequency of the electron is independent of the initial conditions and is the same as the incident frequency of the electromagnetic waves. The refractive index of the medium is

$$n = \frac{c}{\mathrm{v}} \qquad (1.41)$$

where c and $\mathbf{v}$ are the velocity of light in vacuum and in the medium respectively. They are given by

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \quad \text{and} \quad \mathrm{v} = \frac{1}{\sqrt{\varepsilon\mu}} \qquad (1.42)$$

Here ε_0 and ε are the electric permitivities and μ_0 and μ are the magnetic permeabilities of vacuum and medium respectively. In terms of these quantities, assuming $\mu_0 \cong \mu$, we get

$$n = \frac{c}{\mathrm{v}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\in} \qquad (1.43)$$

where *Î* is the relative permittivity of the medium. In general, for the ionosphere, $e < e_0$ and hence from Eq. (1.43), $\mathbf{v} > c$. That is, phase velocity v of radiowaves in the ionosphere is greater than c, the velocity of the radiowaves in vacuum. Also, we see from Eq. (1.43) that the refractive index n of the ionosphere is less than the refractive index $n_0 = 1$ of vacuum. This results in the reflection of the waves from the ionosphere back to earth as shown in Fig. 1.3.

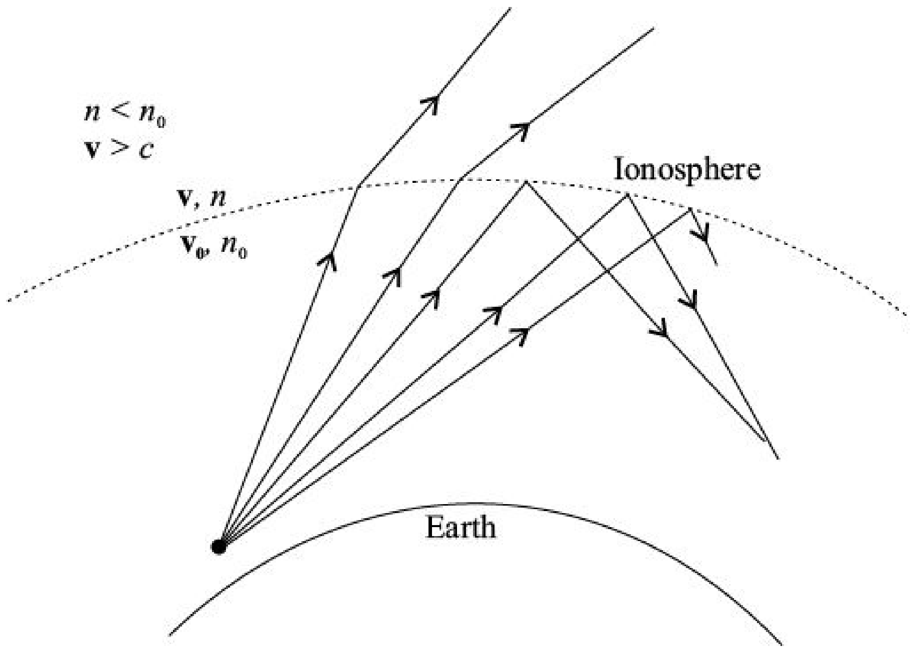


Fig. 1.3 Reflection and refraction of radiowaves by ionosphere.

The oscillating part of *x*, Eq. (1.40), gives rise to an electric dipole moment **p** given by

$$\mathbf{p} = -\frac{e^2}{m\omega^2}\mathbf{E} \tag{1.44}$$

If N is the electron density, the total polarization

$$\mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E} \tag{1.45}$$

which is inversely proportional to w^2 and it changes the refractive index of the ionosphere. If w is very large, the refractive index n of the ionosphere is also very close to unity and the waves are refracted away from the normal. Figure 1.3 also shows the refraction of radiowaves by ionosphere.

1.8 MOTION UNDER A VELOCITY DEPENDENT FORCE

It is very often the case that, in addition to constant applied forces, forces that depend on velocity are present. When a body is falling in a gravitational field, in additional to the gravitational force, there exists a force of resistance offered by air which is dependent on velocity. When bodies move through fluids, the viscous forces oppose the motion. For such systems, Newton's second law may be written in the form

$$m\frac{dv}{dt} = F(v) \tag{1.46}$$

which can be written as

$$m\frac{dv}{dx}\frac{dx}{dt} = F(v) \quad \text{or} \quad mv\frac{dv}{dx} = F(v) \tag{1.47}$$

Equation (1.46) or (1.47) can be solved to analyze the motion. Integration of

Eq. (1.46) gives $t = t_0 + m\int_{v_0}^{v}\frac{dv}{F(v)}$ (1.48)

which gives time as a function of velocity. Here v_0 is the velocity at time $t = t_0$. Integration of Eq. (1.47) gives position as a function of velocity

$$x = x_0 + m\int_{v_0}^{v}\frac{v\,dv}{F(v)} \tag{1.49}$$

1.9 MOTION OF CHARGED PARTICLES IN MAGNETIC FIELDS

Consider a charged particle having a charge *q*, mass *m* and velocity **v** moving in a uniform magnetic field **B**. The force experienced by the charge is given by **F** = *q***v** **B** (1.50) The equation of motion of the particle is

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v}\times\mathbf{B} \tag{1.51}$$

Taking the dot product of Eq. (1.51) with **v**, the right hand side becomes **v . (v** × **B)** = 0. Consequently,

$$m\mathbf{v}.\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}^2\right) = 0 \tag{1.52}$$

That is, the kinetic energy of the particle is a constant.

The velocity $\mathbf{v}$ may be resolved into two components, one parallel to $\mathbf{B}$ (denoted by $\mathbf{v}_{ll}$) and the other perpendicular to $\mathbf{B}$ (denoted by $\mathbf{v}_{\perp}$). Since $\mathbf{v}_{ll}$ ´ $\mathbf{B}$ = 0, Eq. (1.51) takes the form

$$m\frac{d}{dt}(\mathbf{v}_{ll} + \mathbf{v}_{\perp}) = q(\mathbf{v}_{ll} + \mathbf{v}_{\perp}) \times \mathbf{B}$$

$$\frac{d\,\mathbf{v}_{ll}}{dt} + \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B}) \qquad (1.5.3)$$

Equation (1.53) splits into two equations, one describing the motion of the particle parallel to the field and the other describing the motion perpendicular to the field.

$$\frac{d\,\mathbf{v}_{ll}}{dt} = 0 \qquad (1.54)$$

$$\frac{d\,\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B}) \qquad (1.55)$$

The velocity $\mathbf{v}_{ll}$ is constant means that the particle moves with uniform velocity along the direction of $\mathbf{B}$ as shown in Fig. 1.4 (a). The quantity $(d\mathbf{v}_{\perp}/dt)$ is always perpendicular to both $\mathbf{B}$ and $\mathbf{v}_{\perp}$ and therefore the perpendicular component makes the particle travel in a circle as shown in Fig. 1.4(b). For keeping the particle in a circular path, the necessary centripetal force is provided by the force Therefore,

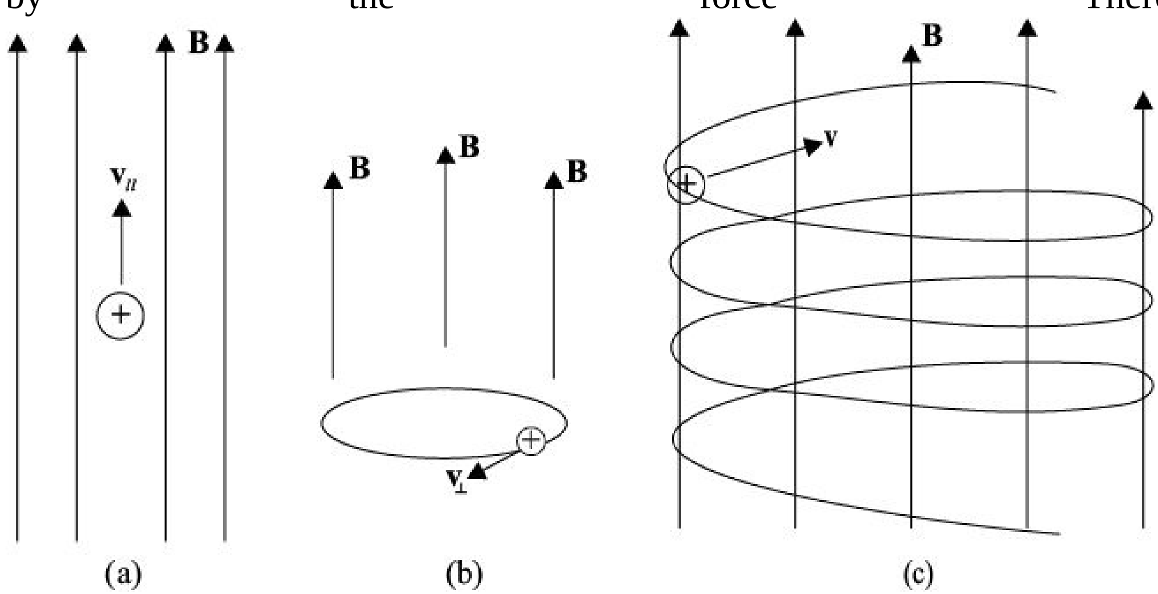


Fig. 1.4 Charged particle motion in a uniform magnetic field.

$$qv_{\perp}B = \frac{mv_{\perp}^2}{R} \qquad (1.56)$$

where R is the radius of the circle. Solving

$$R = \frac{mv_{\perp}}{qB} \qquad (1.57)$$

The radius R is often called the **Larmor radius** of the particle. Period of revolution

$$T = \frac{2\pi R}{v_{\perp}} = \frac{2\pi R}{qBR/m} = \frac{2\pi m}{qB} \qquad (1.58)$$

Angular frequency ω is given by

$$\omega = \frac{2\pi}{T} = \frac{qB}{m} \qquad (1.59)$$

The complete motion of the particle is obtained by combining the two motions, one a uniform motion along the magnetic field line and the other in a circle in a plane perpendicular to the field line. The resulting motion is along a helical path as shown in Fig. 1.4 (c).

WORKED EXAMPLES Example 1.1 Is the force F = A r conservative ?

Solution: (i) **F** = **A** **r**

$$= \hat{i}\,(A_y z - A_z y) + \hat{j}(A_z x - A_x z) + \hat{k}(A_x y - A_y x)$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (A_y z - A_z y) & (A_z x - A_x z) & (A_x y - A_y x) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(A_x y - A_y x) - \frac{\partial}{\partial z}(A_z x - A_x z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(A_y z - A_z y)\right.$$

$$\left. - \frac{\partial}{\partial x}(A_x y - A_y x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(A_z x - A_x z) - \frac{\partial}{\partial y}(A_y z - A_z y)\right] \neq 0$$

Hence, the force is not conservative.

Example 1.2 Find the potential energy function associated with the force $\mathbf{F} = -\,yz\hat{i} - xz\hat{j} - xy\hat{k}$

Solution: It is given that $F_x = yz$, $F_y = -xz$, $F_z = -xy$

$$V = -\int(-yz)dx + f_1(y, z) = xyz + f_1(y, z)$$

$$V = -\int(-xz)dy + f_2(x, z) = xyz + f_2(x, z)$$

$$V = -\int(-xy)\,dz + f_3(x, y) = xyz + f_3(x, y)$$

From these expressions for V, a single consistent equation is: $V = xyz + C$ C = constant **Example 1.3** A particle of mass m moves under a force $\mathbf{F} = -\,cx^3$, where c is a positive constant. (i) Find the potential energy function; (ii) If the particle starts from rest at $x = -\,a$, what is its velocity when it reaches $x = 0$? (iii) Where in the subsequent motion does it come to rest?

Solution: (i) Force $= -\frac{dV}{dx}$ or $-\frac{dV}{dx} = -cx^3$

$$dV = cx^3 dx \quad \text{or} \quad V = \frac{cx^4}{4} + \text{constant}$$

(ii) Work done $W = \int_{-a}^{0} F\, dx = \frac{1}{2} m\text{v}^2$

$$\int_{-a}^{0} (-cx^3) dx = \frac{1}{2} m\text{v}^2 \quad \text{or} \quad \frac{ca^4}{4} = \frac{1}{2} m\text{v}^2$$

$$\text{v} = \sqrt{\frac{c}{2m}} a^2$$

(iii) Let at $x = b$ it comes to rest.

$$\int_{-a}^{b} (-cx^3)\, dx = T_b - T_{-a} = 0 \quad \text{or} \quad -\frac{cb^4}{4} + \frac{ca^4}{4} = 0$$

or $$b = \pm a$$

Again, it comes to rest at $x = +a$.

Example 1.4 An artificial satellite is placed in an elliptical orbit about the earth. Its point of closest approach (perigee) is at a distance r_p from the centre of the earth, while its point of greatest distance (apogee) is at a distance r_a from the centre of the earth. If the speed of the satellite at the perigee is v_p, find the speed at the apogee.

Solution: The gravitational force on the satellite, not a very significant one, exerts no torque as the force passes through the axis of rotation. Hence, the angular $I_a \omega_a = I_p \omega_p \quad \omega_a = \frac{\text{v}_a}{r_a}$ and $\omega_p = \frac{\text{v}_p}{r_p}$

momentum of the satellite is constant at all times: where I_a and I_p are the M.I. of the satellite about the axis of rotation when it is at apogee and perigee respectively. The orbiting satellite can be considered as a point mass and

$$I_a = mr_a^2 \quad \text{and} \quad I_p = mr_p^2$$

With these values

therefore
$$mr_a^2 \times \frac{v_a}{r_a} = mr_p^2 \times \frac{v_p}{r_p}$$

$$v_a = \frac{r_p v_p}{r_a}$$

which is independent of the mass of the satellite.

Example 1.5 A particle of mass *m* is projected vertically up with an initial velocity of v_0. If the force due to the friction of the air is directly proportional to its instantaneous velocity, calculate the velocity and position of the particle as a function of time.

Solution: For the particle moving up the frictional force is downward. Hence, the total force acting on the particle

$$F = -mg - k\mathbf{v} \qquad k = \text{constant}$$

$$t = \int_{v_0}^{v} \frac{m\,dv}{F(v)} = -m\int_{v_0}^{v} \frac{dv}{mg + kv}$$

$$t = -\frac{m}{k} ln\,(mg + kv)\Big|_{v_0}^{v} = \frac{m}{k} ln\left(\frac{mg + kv_0}{mg \quad kv}\right)$$

$$\frac{mg + kv}{mg + kv_0} = e^{-(k/m)t}$$

$$v = \left(\frac{mg}{k} + v_0\right)e^{-(k/m)t} - \frac{mg}{k}$$

Since $\mathbf{v} = dx/dt$

$$dx = \left[\left(\frac{mg}{k} + v_0\right)e^{-(k/m)t} - \frac{mg}{k}\right]dt$$

$$x = x_0 - \frac{mgt}{k} + \left(\frac{m^2 g}{k^2} + \frac{mv_0}{k}\right)[1 - e^{-(k/m)t}]$$

Example 1.6 A mass *m* tied to a spring having a force constant k oscillates in one dimension. If the motion is subjected to the force $F = -kx$, find expressions for displacement, velocity and period of oscillation.

Solution: $$F = -kx, V(x) = -\int_0^x (-kx)\, dx = \frac{1}{2}kx^2 \qquad \text{(i)}$$

Conservation of energy gives

$$\frac{1}{2}m\text{v}^2 + \frac{1}{2}kx^2 = E \qquad \text{(ii)}$$

$$\text{v}^2 = \left(\frac{dx}{dt}\right)^2 = \frac{2E}{m}\left(1 - \frac{k}{2E}x^2\right) \qquad \text{(iii)}$$

Defining

$$\sqrt{\frac{k}{2E}}\,x = \sin\theta \quad dx = \sqrt{\frac{2E}{k}}\cos\theta\, d\theta \qquad \text{(iv)}$$

Substituting Eq. (iv) in (iii)

$$\left(\frac{dx}{dt}\right)^2 = \frac{2E}{m}\cos^2\theta \qquad \text{(v)}$$

$$dt = \frac{dx}{\sqrt{(2E/m)}\cos\theta} = \sqrt{\frac{m}{k}}d\theta \qquad \text{(vi)}$$

Writing $\omega = \sqrt{k/m}$ and integrating

$$t = \frac{\theta - \theta_0}{\omega} \quad \text{or} \quad \theta = \omega t + \theta_0$$

Substituting the value of θ in Eq. (iv)

$$x = \sqrt{\frac{2E}{k}}\sin(\omega t + \theta_0) \qquad \text{(vii)}$$

It is obvious that the motion is simple harmonic with amplitude $\sqrt{2E/k}$ and angular frequency $\omega = \sqrt{k/m}$.

$$\text{Period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{(viii)}$$

$$\text{Velocity } \text{v} = \frac{dx}{dt} = \sqrt{\frac{2E}{k}}\,\omega\cos(\omega t + \theta_0) \qquad \text{(ix)}$$

Example 1.7 A particle of mass *m* is at rest at the origin of the co-ordinate system. At $t = 0$, a force $F = F_0 (1 - te^{-lt})$ is applied to the particle. Find the velocity and position of the particle as a function of time.

Solution: By Newton's second law

$$m\frac{dv}{dt} = F_0(1 - te^{-\lambda t}) \quad \text{or} \quad dv = \frac{F_0}{m}(1 - te^{-\lambda t})\,dt$$

Integrating between the limits $t = 0$ and $t = t$

$$v = \frac{F_0 t}{m} - \frac{F_0}{m}\int_0^t te^{-\lambda t}\,dt$$

Integration by parts lead to

$$v = \frac{F_0 t}{m} + \frac{F_0}{m\lambda}\left[te^{-\lambda t} + \frac{1}{\lambda}(e^{-\lambda t} - 1)\right]$$

Since $v = dx/dt$

$$dx = \left[\frac{F_0 t}{m} + \frac{F_0}{m\lambda}\left(te^{-\lambda t} + \frac{e^{-\lambda t}}{\lambda} - \frac{1}{\lambda}\right)\right]dt$$

$$x = \frac{F_0}{m}\frac{t^2}{2} - \frac{F_0 t}{m\lambda^2} + \frac{F_0}{m\lambda^2}\left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^t + \frac{F_0}{m\lambda}\left[-\frac{te^{-\lambda t}}{\lambda} - \frac{e^{-\lambda t}}{\lambda^2}\right]_0^t$$

$$= \frac{F_0 t^2}{2m} - \frac{F_0 t}{m\lambda^2} + \frac{F_0}{m\lambda^2}\left(\frac{e^{-\lambda t}}{-\lambda} + \frac{1}{\lambda}\right) + \frac{F_0}{m\lambda}\left[\frac{-te^{-\lambda t}}{\lambda} - \frac{e^{-\lambda t}}{\lambda^2} + \frac{1}{\lambda^2}\right]$$

$$= \frac{2F_0}{m\lambda^3}(1 - e^{-\lambda t}) - \frac{F_0 t}{m\lambda^2}(1 + e^{-\lambda t}) + \frac{F_0}{2m}t^2$$

Example 1.8 A particle having total energy *E* is moving in a potential field $V(r)$. Show that the time taken by the particle to move from r_1 to r_2 is

$$t_2 - t_1 = \int_{t_1}^{t_2} \frac{dr}{\sqrt{2(E - V)/m}}$$

Solution: The particle is moving under the action of a position-dependent force and therefore the sum of its kinetic and potential energies is *E*. That is,

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V(r) = E$$

$$\left(\frac{dr}{dt}\right)^2 = 2\frac{(E-V)}{m} \quad \text{or} \quad \frac{dr}{dt} = \sqrt{\frac{2(E-V)}{m}}$$

$$dt = \frac{dr}{\sqrt{2(E-V)/m}}$$

Integrating,

$$t_2 - t_1 = \int_{t_1}^{t_2} \frac{dr}{\sqrt{2(E-V)/m}}$$

Example 1.9 A disc of mass *m* and radius *r* rolls down an inclined plane of angle *q*. Find the acceleration of the disc and the frictional force.

Solution: Forces acting on the disc are the weight *mg*, the reaction *R* and the frictional force *f*. Let the acceleration of the disc be *a*. (See Fig. 1.5.)

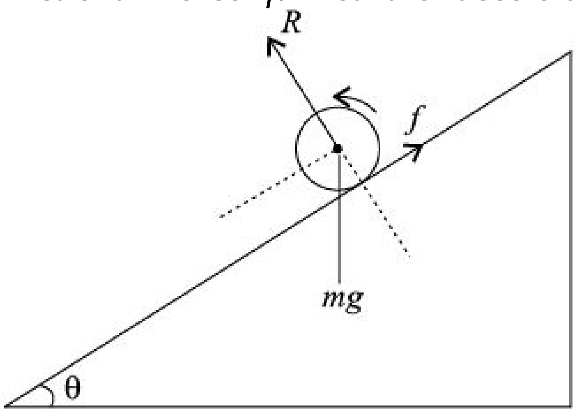


Fig. 1.5 Disc rolling down a plane.

The unbalanced force on the disc = *mg* sin*q* – *f* This must be equal to *ma*. Hence, *ma* = *mg* sin*q* – *f* Moment of inertia of the disc about its point of contact

$$I = \frac{mr^2}{2} + mr^2 = \frac{3}{2}mr^2$$

Torque acting,

$$\tau = \frac{dL}{dt} = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = \frac{3}{2}mr^2\dot{\omega}$$

Taking moment about the point of contact

$$\tau = \mathbf{r} \times \mathbf{F} = r\,mg\,\sin\theta \sin 90 = r\,mg\,\sin\theta$$

Equating the two expressions for torque

$$\frac{3}{2}mr^2\dot{\omega} = rmg\sin\theta \quad or \quad \dot{\omega} = \frac{2}{3}g\frac{\sin\theta}{r}$$

Acceleration $a = r\dot{\omega} = \frac{2}{3}g\sin\theta$

Frictional force $f = m\,g\,\sin\theta - ma = m\,g\,\sin\theta - m\frac{2}{3}g\,\sin\theta = \frac{1}{3}mg\,\sin\theta$

Example 1.10 Consider a body of mass *m* projected with velocity v_0 at an angle a with the horizontal. Derive expressions for the range and time of flight of the projectile.

Solution: The motion remains in the vertical plane containing the velocity vector v_0. The horizontal may be taken as the *x*-axis and the vertical in the plane of motion may be taken as the *y*-axis. The equations of motion of the projectile are

$$m\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad m\frac{d^2y}{dt^2} = -mg \tag{i}$$

These are subjected to the initial conditions:

$$x(t = 0) = 0 \qquad y(t = 0) = 0$$

$$\dot{x}(t = 0) = \mathrm{v}_0 \cos\alpha \quad \dot{y}(t = 0) = \mathrm{v}_0 \sin\alpha \tag{ii}$$

Integration of the equations of motion gives

$$\frac{dx}{dt} = c_1 \quad \frac{dy}{dt} = -gt + c_2 \tag{iii}$$

where c_1 and c_2 are constants. Applying the initial conditions

$$\frac{dx}{dt} = \mathrm{v}_0 \cos\alpha \qquad = -gt + \mathrm{v}_0 \sin\alpha$$

Integrating again

$$x = \mathrm{v}_0 t \cos\alpha \qquad y = -\frac{1}{2}gt^2 + \mathrm{v}_0 t \sin\alpha \tag{iv}$$

Substituting the value of t from the expression for x

$$y = -\frac{1}{2}g\frac{x^2}{\mathbf{v}_0^2\cos^2\alpha} + \mathbf{v}_0 \sin\alpha \frac{x}{\mathbf{v}_0 \cos\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2\mathbf{v}_0^2}(1 + \tan^2\alpha)$$

This represents the trajectory of a parabola. The range R of the projectile is obtained by putting $x = R$ and $y = 0$.

$$0 = R\tan\alpha - \frac{gR^2}{2\mathrm{v}_0^2}(1 + \tan^2\alpha)$$

$$R = \frac{2\mathrm{v}_0^2}{g}\frac{\tan\alpha}{1 + \tan^2\alpha} = \frac{2\mathrm{v}_0^2}{g}\frac{\sin\alpha}{\cos\alpha}\frac{1}{\sec^2\alpha} = \frac{\mathrm{v}_0^2}{g}\sin 2\alpha$$

The time of flight $\qquad T = \frac{R}{\mathrm{v}_0\cos\alpha} = \frac{2\mathrm{v}_0}{g}\sin\alpha$

REVIEW QUESTIONS 1. What are inertial and non-inertial

frames of references?

2. Is inertial mass same as gravitational mass? Explain.
3. Under what condition do we write Newton's second law in the form $\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt}$?
4. "Law of conservation of linear momentum is a consequence of Newton's first law". Substantiate.
5. When do you say a force is conservative? Illustrate with an example.
6. State and explain the analogue of Newton's second law in rotational motion.
7. Explain angular momentum conservation by taking an earth satellite as an example.
8. "Absolute value of the potential and kinetic energies has no meaning." Comment.
9. For a particle moving under the action of a force, prove that the sum of its kinetic energy and potential energy remains constant throughout its motion.
10. A force **F** acts on a particle, giving it displacement $d\mathbf{r}$. If $\mathbf{F}.d\mathbf{r} = -dV(r)$, where $V(r)$ is a scalar function of r, show that $\mathbf{F} = -\nabla V(r)$
11. How are refraction and reflection by the ionosphere possible?
12. Consider a particle having a charge q, mass m and velocity **v** moving in a uniform magnetic field **B**. Explain the resulting motion of the charged particle.

PROBLEMS

1. Find the potential energy function that corresponds to the force $\mathbf{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$
2. Find whether the following force is conservative, if so find the corresponding potential function: $\mathbf{F} = (ax + by^2)\hat{i} + (az + 2bxy)\hat{j} + (ay + bz^2)\hat{k}$, where a, b are constants.
3. A ladder of length $2l$ and mass m is resting against a wall. Coefficient of friction between the ladder and the wall is m and that between the ladder and the horizontal floor is m. The ladder makes an angle q with the horizontal. When the ladder is about to slip, show that $\tan\theta = \frac{1 - \mu\mu'}{2\mu}$
4. A block of mass m is at rest on a frictionless surface at the origin. At time $t = 0$, a force $F = F_0 e^{-lt}$ = where l is a small positive constant, is applied.

Calculate $x(t)$ and v(t).

5. A particle of mass m is falling under the action of gravity near the surface of the earth. If the force due to the friction of the air is directly proportional to its instantaneous velocity, calculate the velocity and position of the mass as a function of time.
6. A particle of mass m having an initial velocity $\mathbf{v}_0$ is subjected to a retarding force proportional to its instantaneous velocity. Calculate its velocity and position as a function of time.
7. A ball of mass m is thrown with velocity $\mathbf{v}_0$ on a horizontal surface, where the retarding force is proportional to the square root of the instantaneous velocity. Calculate the velocity and position of the ball as a function of time.
8. A particle of mass m is at rest at $t = 0$ when it is subjected to a force $F = A \sin wt$. Calculate the values of $x(t)$ and v(t).
9. A particle of mass m is at rest at the origin of the co-ordinate system. At $t = 0$, a force bt starts acting on the particle. Find the velocity and position of the particle as a function of time.
10. The components of a force acting on a particle are $F_x = ax + by^2$, $F_y = az + 2bxy$ and $F_z = ay + bz^2$, where a and b are constants. Evaluate the work done in taking the particle from the origin to the point (1, 1, 0) by moving first along the x-axis and then parallel to the y-axis.
11. A particle of mass m moves in a central force field $V(r) = kmr^3$ ($k > 0$). If its path is a circle of radius a , then (i) what is its period? (ii) what is its angular momentum?
12. A particle having a charge q, mass m and velocity $\mathbf{v}$ is moving in a uniform magnetic field $\mathbf{B}$. If the field is perpendicular to $\mathbf{v}$, prove that the kinetic energy of the particle is a constant.
13. A particle of mass m moves along the x-axis under a constant force f starting from rest at the origin at time $t = 0$. If T and V are the kinetic and potential energies of the particle, calculate $\int_0^{t_0} (T - V)dt$.

2

System of Particles

The mechanics of a system of particles can be studied by using a straightforward application of Newton's laws.This application of Newton's laws considers the forces acting between particles in addition to the externally applied forces. One can easily extend the considerations of the mechanics of a single particle to a system of particles also.

2.1 CENTRE OF MASS

The mass of a point particle is concentrated at a particular point. When we consider the motion of a system of n particles, there is a point in it which behaves as if the entire mass of the system is concentrated at that point. This point is called the **centre of mass** of the system. The centre of mass C of a system of particles

(see Fig. 2.1) whose radius vector is $\mathbf{R}$ is related to the masses m_i and radius vectors $\mathbf{r}_i$ of all n particles of the system by the equation

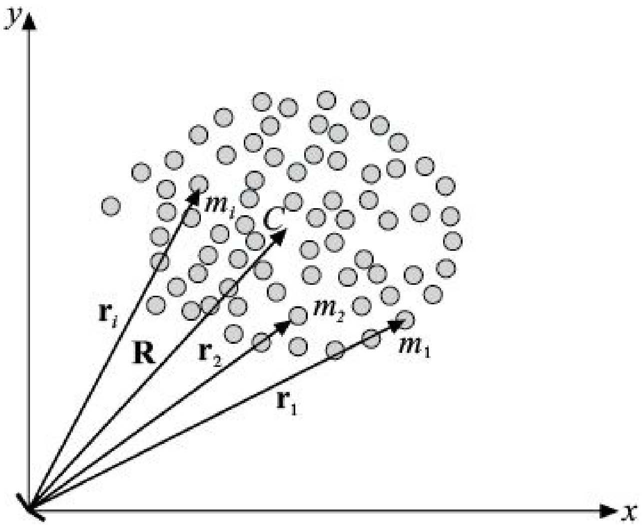


Fig. 2.1 Centre of mass of a system of *n* particles.

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_i\mathbf{r}_i + \ldots}{m_1 + m_2 + \ldots + m_i + \ldots} = \frac{\sum_i m_i\mathbf{r}_i}{M} \qquad (2.1)$$

where *M* is the total mass of the system. For a continuous body, the co-ordinates of the centre of mass are

$$X = \frac{1}{M}\int_V \rho x\, dV \qquad Y = \frac{1}{M}\int_V \rho y\, dV \qquad Z = \frac{1}{M}\int_V \rho z\, dV$$

For a homogeneous body, the density ρ is constant and *V* is the volume of the body

$$X = \frac{1}{V}\int_V x\, dV \qquad Y = \frac{1}{V}\int_V y\, dV \qquad Z = \frac{1}{V}\int_V z\, dV \qquad (2.1a)$$

A frame of reference with the centre of mass as the origin is called the **centre of mass frame of reference**. In this frame of reference, obviously, the position vector of the centre of mass **R** is equal to zero. Consequently, the linear momentum **P** of the system ($d\mathbf{R}/dt$) is also zero. It is the practice to deal with all scattering problems in nuclear physics in this frame of reference.

2.2 CONSERVATION OF LINEAR MOMENTUM

Consider a system of *n* particles of masses $m_1, m_2, m_3, \ldots m_n$. Let their position vectors at time *t* be $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots \mathbf{r}_n$. The force acting on the *i*th particle $\mathbf{F}_i$ has two parts: (i) a force applied on the system from outside or external force (ii) an internal force which is a force among the particles of the system. Newton's second law for the *i*th particle of the system can be written as

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i = \mathbf{F}_{ie} + \sum_{j=1}^{n} \mathbf{F}_{ij} \quad j \neq i \qquad (2.2)$$

where $\mathbf{F}_{ie}$ is the external force on the *i*th particle and $\mathbf{F}_{ij}$ is the internal force on the *i*th particle due to the *j*th one. Since $\mathbf{F}_{ii} = 0$, *j* π *i* in the summation. Summing over all particles of the system, Eq. (2.2) takes the form

$$\sum_{i=1}^{n}\frac{d\,\mathbf{p}_i}{dt}=\sum_{i=1}^{n}\mathbf{F}_{ie}+\sum_{i=1}^{n}\ \sum_{j=1}^{n}\mathbf{F}_{ij}\qquad j\neq i \qquad (2.3)$$

Assuming that Newton's third law is valid for the internal force $\mathbf{F}_{ij}= -\mathbf{F}_{ji}$ (2.4) Use of this condition reduces the second term on the right of Eq. (2.3) to zero. The first term $\sum_i \mathbf{F}_{ie}=\mathbf{F}_e$, the total external force acting on the system. The sum $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + ... + \mathbf{p}_n = \mathbf{P}$
is the total linear momentum of the system. Now Eq. (2.3) reduces to

$$\frac{d\,\mathbf{P}}{dt}=\mathbf{F}_e \qquad (2.5)$$

which provides the law of conservation of linear momentum of a system of particles: *If the external force acting on a system of particles is zero, then the total linear momentum of the system is conserved.*
When external force acting on a system is zero, it is called a ***closed system***. For a closed system, linear momentum is conserved.

Another interesting result is the relation connecting the total linear momentum and the velocity of the centre of mass. With the definition of centre of mass in Eq. (2.1)

$$\sum_i\frac{d\,\mathbf{p}_i}{dt}=\frac{d^2}{dt^2}\sum_i m_i\mathbf{r}_i=M\,\frac{d^2\mathbf{R}}{dt^2}$$

and Eq. (2.3) takes the form

$$M\,\frac{d^2\mathbf{R}}{dt^2}=\frac{d\,\mathbf{P}}{dt}=\mathbf{F}_e \qquad (2.6)$$

That is, the centre of mass moves as if the total external force were acting on the entire mass of the system concentrated at the centre of mass.

2.3 ANGULAR MOMENTUM

We now derive the angular momentum **L** of a system of particles which is defined as $\mathbf{L} = \sum_i \mathbf{r_i} \times \mathbf{p}_i$ (2.7)

Figure 2.2 illustrates the position vector of the centre of mass of the system and that of the *i*th particle.

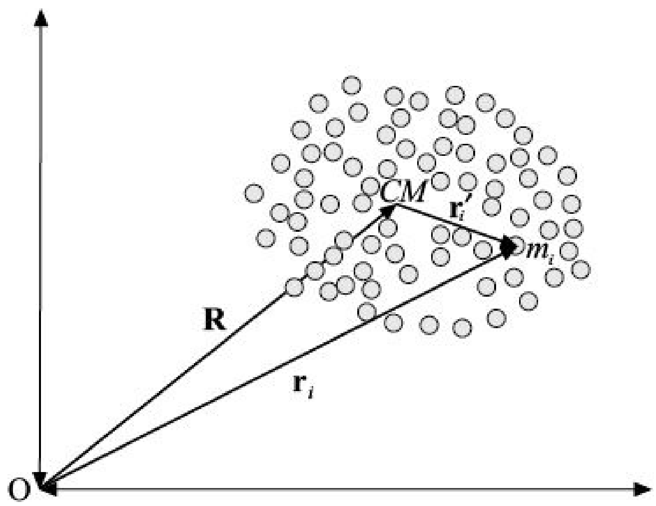


Fig. 2.2 Position of centre of mass and *i*th particle.

From Fig. 2.2 we have

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i' \quad \text{or} \quad \dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}_i' \tag{2.8}$$

Consequently,

$$\mathbf{p}_i = m_i \dot{r}_i = m_i (\dot{\mathbf{R}} + \dot{\mathbf{r}}_\mathbf{i}') \tag{2.9}$$

Substituting these values in Eq. (2.7)

$$L = \sum_i (R + r_i') \times m_i \, (\dot{\mathbf{R}} + \dot{r}_i')$$

$$= \sum_i m_i \, (\mathbf{R} \times \dot{\mathbf{R}}) + \left(\mathbf{R} \times \sum_i m_i \, \dot{r}'_i \right) + \left(\sum_i m_i \mathbf{r}_i' \times \dot{\mathbf{R}} \right) + \sum_i (m_i \mathbf{r}_i' \times \dot{r}'_i) \tag{2.10}$$

The quantity $\sum_i m_i r_i'$ vanishes as it defines the radius vector of the centre of mass in the co-ordinate system in which the origin is the centre of mass. The quantity

$$\sum_i m_i \dot{\mathbf{r}}_i' = \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i'\right) = 0$$

Hence, the total angular momentum

$$\mathbf{L} = \sum_i m_i \,(\mathbf{R} \times \dot{\mathbf{R}}) + \sum_i m_i r_i' \times \dot{r}_i$$

$$= \mathbf{R} \times M\mathbf{V}_{CM} + \sum_i (\mathbf{r}_i' \times \mathbf{p}_i') \qquad (2.11)$$

where $\mathbf{V}_{CM}$ is the velocity of the centre of mass with respect to the origin O. The meaning of the equation is that the total angular momentum about a point O is equal to the sum of the angular momentum of the system concentrated at the centre of mass and the angular momentum of the system of particles about the centre of mass.

2.4 CONSERVATION OF ANGULAR MOMENTUM

We now consider the angular momentum of a system of n particles which is

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i$$

defined as

$$\frac{d\mathbf{L}}{dt} = \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \qquad (2.12)$$

The first term on the right is zero since the vector product of a vector with itself is zero. Substituting for $(d\mathbf{p}_i / dt)$ from Eq. (2.2)

$$\frac{d\mathbf{L}}{dt} = \sum_i \left[\mathbf{r}_i \times \left(\mathbf{F}_{ie} + \sum_j \mathbf{F}_{ij}\right)\right]$$

$$= \sum_i (\mathbf{r}_i \times \mathbf{F}_{ie}) + \sum_i \sum_j (\mathbf{r}_i \times \mathbf{F}_{ij}) \quad j \neq i \qquad (2.13)$$

The second term on the right contains pairs of terms like

$$\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}$$

Since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$,this pair reduces to

$$\mathbf{r}_i \times \mathbf{F}_{ij} - \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{r}_{ij} \times \mathbf{F}_{ij} \qquad (2.14)$$

which is zero if the internal forces are central, that is, the internal forces are along the line joining the two particles. Hence, the second term on the right of Eq. (2.13) vanishes. Since $\mathbf{r}_i \ \mathbf{F}_{ie}$ is the torque due to the external force on the *i*th particle, Eq. (2.13) reduces to

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{N}_{ie} = \mathbf{N}_e \qquad (2.15)$$

where N_e is the total external torque acting on the system. Eq. (2.15) leads to the conservation law: *If the total torque due to external forces on a system of particles is zero, then the total angular momentum is a constant of motion.*

2.5 KINETIC ENERGY FOR A SYSTEM OF PARTICLES

For a system of particles the kinetic energy of the system

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \mathbf{v}_i^2 \qquad (2.16)$$

The position of the centre of mass of the system and that of the *i*th particle is shown in Fig. 2.2. From the figure, we have

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i' \quad \text{or} \quad \mathbf{v}_i = \mathbf{V}_{CM} + \mathbf{v}_i'$$

With this value of $\mathbf{v}_i$, Eq. (2.16) takes the form

$$T = \frac{1}{2}\sum_i m_i\, (\mathbf{V}_{CM} + \mathbf{v}_i').(\mathbf{V}_{CM} + \mathbf{v}_i')$$

The term $\sum m_i \mathbf{r}_i'$ vanishes as it defines the radius vector of the centre of mass in the co-ordinate system in which the origin is the centre of mass. Hence,

$$T = \frac{1}{2} M\, \mathbf{V}_{CM}^2 + \frac{1}{2}\sum_i m_i\, \mathbf{v}_i'^2 \qquad (2.18)$$

Thus, like angular momentum, the kinetic energy also consists of two parts:

(i) the kinetic energy obtained if all the mass were concentrated at the centre of mass, and (ii) the kinetic energy of motion about the centre of mass.

2.6 ENERGY CONSERVATION OF A SYSTEM OF PARTICLES

The energy conservation law of a single particle system can easily be extended to a system of particles. The force acting on the ith particle is given by Eq. (2.2). As in the case of a single particle, the work done by all forces in moving the system from an initial position 1 to a final position 2 is given by

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \, . d\mathbf{r}_i = \sum_i \int_1^2 \mathbf{F}_{ie} \, . d\mathbf{r}_i + \sum_i \sum_j \int_1^2 \mathbf{F}_{ij} \, . d\mathbf{r}_i \quad i \neq j \qquad (2.19)$$

Again, reducing the integral $\sum_i \int_1^2 \mathbf{F}_i \, . d\mathbf{r}_i$ using equation of motion, we have

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \, . d\mathbf{r}_i = \sum_i m_i \int_1^2 \frac{d\mathbf{v}_i}{dt} . d\mathbf{r}_i = \sum_i m_i \int_1^2 \frac{d\mathbf{v}_i}{dt} . \mathbf{v}_i dt$$

$$= \sum_i m_i \int_1^2 \mathbf{v}_i \, d\mathbf{v}_i = \sum_i \frac{1}{2} m_i [\mathbf{v}_i^2]_1^2$$

$$= \mathrm{T}_2 - \mathrm{T}_1 \qquad (2.20)$$

where T is the total kinetic energy of the system.

Next we consider the right hand side of Eq. (2.19). If both $\mathbf{F}_{ie}$ and $\mathbf{F}_{ij}$ are conservative, they are derivable from potential functions $\mathbf{F}_{ie} = -\nabla_i V_{ie}(\mathbf{r}_i)$ and $\mathbf{F}_{ij} = -\nabla V_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ (2.21) where the subscript i on the del operator indicates that the derivative is with respect to the co-ordinates of the ith particle. The first term on the right side of Eq. (2.19) now takes the form

$$\sum_i \int_1^2 \mathbf{F}_{ie} \cdot d\mathbf{r}_i = -\sum_i \int_1^2 \nabla_i V_{ie} (\mathbf{r}_i) \cdot d\mathbf{r}_i$$

$$= -\sum_i \int_1^2 dV_{ie} = -\left[\sum_i V_{ie}\right]_1^2 \qquad (2.22)$$

In order to satisfy Newton's third law, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. Hence,

$$\sum_i \sum_j \mathbf{F}_{ij} \cdot d\mathbf{r}_i = \sum_i \sum_j \mathbf{F}_{ji} \cdot d\mathbf{r}_j = -\sum_i \sum_j \mathbf{F}_{ij} \cdot d\mathbf{r}_j \quad i \neq j \qquad (2.23)$$

Consequently,

$$\sum_i \sum_j \mathbf{F}_{ij} \cdot d\mathbf{r}_i = \frac{1}{2} \sum_i \sum_j \mathbf{F}_{ij} \cdot (d\mathbf{r}_i - d\mathbf{r}_j) \quad i \neq j$$

$$= \frac{1}{2} \sum_i \sum_j \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \qquad d\mathbf{r}_{ij} = d\mathbf{r}_i - d\mathbf{r}_j \quad i \neq j \qquad (2.24)$$

where the factor ½ is introduced to avoid each member of a pair being included twice, first in the i summation and then in the j summation. Substituting this value

$$\sum_i \sum_j \int_1^2 \mathbf{F}_{ij} \cdot d\mathbf{r}_i = \frac{1}{2} \sum_i \sum_j \int_1^2 \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \quad i \neq j$$

$$= -\frac{1}{2} \sum_i \sum_j \int_1^2 \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij} \quad i \neq j$$

$$= -\frac{1}{2} \sum_i \sum_j \int_1^2 d\mathbf{V}_{ij}$$

$$= -\frac{1}{2} \left[\sum_i \sum_j V_{ij}\right]_1^2 \quad i \neq j \qquad (2.25)$$

Here ∇_{ij} stands for the gradient with respect to r_{ij}.

Equation (2.19) can now be written as

$$W_{12} = -\left[\sum_i V_{ie}\right]_1^2 - \frac{1}{2}\left[\sum_i \sum_j \mathbf{V}_{ij}\right]_1^2 \quad i \neq j \qquad (2.26)$$

As the internal and external forces are derivable from potentials, it is possible to define a total potential energy *V* of the system:

$$V = \sum_i V_{ie} + \frac{1}{2}\sum_i \sum_j V_{ij} \quad i \neq j \qquad (2.27)$$

With this potential, Eq. (2.19) reduces to $W_{12} = -(V_2 - V_1)$ (2.28) From Eqs. (2.20) and (2.28), we get $T_2 - T_1 = V_1 - V_2$

$$T_1 + V_1 = T_2 + V_2 \qquad (2.29)$$

which gives the energy conservation law: *For a conservative system of n particles, the total energy E = T + V is constant*, where T is given by Eq. (2.18) and *V* by Eq. (2.27).

2.7 TIME VARYING MASS SYSTEMS—ROCKETS

So far we have been studying systems in which the mass is constant. We shall now investigate a system in which the mass is time-varying. The time variation of the mass in a rocket is due to the expulsion of the exhaust. The mass of an object can also vary due to its very high speed (relativistic effect) which is different from a time-varying mass system.

Consider a rocket which is propelled in a forward direction by the ejection of mass exhaust in the backward direction in the form of gases resulting from the combustion of fuel. Thus, the forward force on the rocket is the reaction to the backward force of the ejected gases. Our aim is to find the velocity of the rocket at any time after take-off from the ground. At time *t* assume that the rocket of mass *m* is moving with a velocity **v** relative to the fixed co-ordinate system, say earth. The exhaust is ejected with a constant velocity **u** relative to the rocket and therefore **v** + **u** relative to the fixed co-ordinate system
(see Fig. 2.3). At time *t* + *dt* the mass of the rocket has changed to *m* + *dm* and the velocity to **v** + *d***v**. At this time an amount of fuel denoted by – *dm* is moving with velocity **v** + **u** relative to the fixed co-ordinate frame.

Momentum of the system at time *t* is $\mathbf{P}(t) = m\mathbf{v}$ Momentum of the rocket alone at *t* + *dt*

$$\mathbf{P}_{rocket}(t + dt) = (m + dm)(\mathbf{v} + d\mathbf{v})$$
$$\cong m\mathbf{v} + md\mathbf{v} + \mathbf{v}dm$$

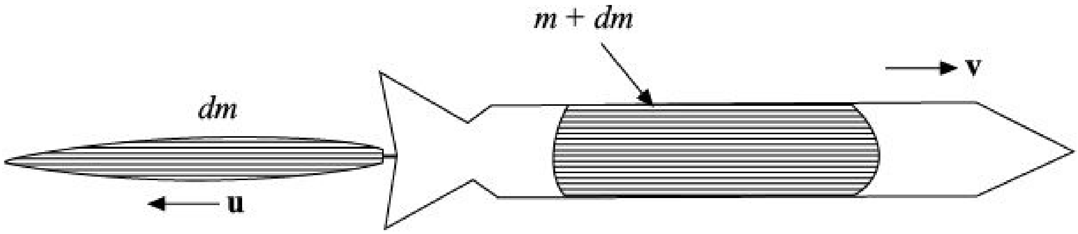


Fig. 2.3 Motion of a rocket at some instant of time.

The second order term $dm\, d\mathbf{v}$ is neglected.

The momentum of the fuel $\mathbf{P}_{\text{fuel}}$ $(t + dt) = -\,dm\,(\mathbf{v} + \mathbf{u}) = -\,dm\mathbf{v} - dm\mathbf{u}$ Hence, the total momentum of the system at $(t + dt)$ is $\mathbf{P}(t + dt) = \mathbf{P}_{rocket}\,(t + dt) + \mathbf{P}_{fuel}\,(t + dt) = m\,\mathbf{v} + md\mathbf{v} - dm\,\mathbf{u}$ (2.30) Change of momentum dP is given by $d\mathbf{P} = \mathbf{P}(t + dt) - \mathbf{P}(t) = md\mathbf{v} - dm\mathbf{u}$ (2.31) Rate of change of momentum

$$\frac{d\mathbf{P}}{dt} = m\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dm}{dt} \tag{2.32}$$

Rate of change of momentum is the external force applied, that is, **F**. Then

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dm}{dt} \tag{2.33}$$

When a rocket is in deep space, $\mathbf{F} = 0$, then

$$m\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dm}{dt} \tag{2.34}$$

The term on the left side, m $(d\mathbf{v}/dt)$, is the thrust, the force felt by the rocket. Thus, the thrust depends on the exhaust velocity **u** and the fuel mass flow rate (dm/dt):

The thrust on the rocket $$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dm}{dt} \tag{2.34a}$$

The solution of Eq. (2.34) is simple. It can be written as

$$d\mathbf{v} = \mathbf{u}\frac{dm}{m}$$

Integrating,

$$\int_{\mathrm{v}_0}^{\mathrm{v}} d\mathbf{v} = \mathbf{u}\int_{m_0}^{m} \frac{dm}{m}$$

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{u}\, ln \frac{m}{m_0} \tag{2.35}$$

Since $m_0 > m$, it is more convenient to write this equation as

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{u}\, ln \frac{m_0}{m} \tag{2.36}$$

where m_0 is the original mass, m is the final mass and $\mathbf{v}_0$ is the initial velocity, velocities **v** and **u** are in opposite directions. Very high final velocity requires large values for the exhaust velocity **u** and large values for m_0/m. Large values of m_0/m can be achieved by reducing m, the final mass which consists of the rocket structure and the payload. To reduce m, **staged rockets** are used. The structure of the first stage is usually very massive as it contains all the necessary fuel, engine, and so on. When its fuel is over, all this structure is jettisoned from the rest of the rocket, so that the entire force is applied to accelerate a much smaller mass.

Near the earth surface, the external force on the rocket due to the attraction of the earth has to be taken into account. In such a situation, Eq. (2.34) takes the form

$$m\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dm}{dt} = m\mathbf{g}$$

$$d\mathbf{v} = \mathbf{u}\frac{dm}{m} + \mathbf{g}dt \tag{2.37}$$

Integrating,

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{u} ln \frac{m_0}{m} + \mathbf{g}t \tag{2.38}$$

Remembering that **u** and **g** are in a direction opposite to that of **v**, the corresponding scalar equation for a rocket fired vertically upward from rest

($\mathbf{v}_0 = 0$) $\mathrm{v} = u\, ln \frac{m_0}{m} - gt$ (2.39) In the present-day rockets, the high final speed is achieved by continued acceleration; the value of the acceleration increases as the remaining mass of the rocket decreases.

Another useful relation is the one connecting the original mass of the rocket m_0, mass of the rocket at time t and the rate of mass decrease a. From definition

$$\frac{dm}{dt} = -\alpha$$

where the negative sign indicates that there is a mass decrease. Integrating,

$$\int dm = -\int \alpha \, dt \quad \text{or} \quad m - m_0 = -\alpha t$$

$$m = m_0 - \alpha t \tag{2.40}$$

The mass at the end of the mission will be the sum of the body of the rocket plus the mass of the satellite or bomb in the case of missiles.

WORKED EXAMPLES

Example 2.1 A body of mass m splits into two masses m_1 and m_2 by an explosion. After the split the bodies move with a total kinetic energy T in the same direction. Show that their relative speed is $\sqrt{2Tm / m_1 m_2}$.

Solution: The initial momentum of the mass is zero. Hence, by the law of conservation of linear momentum

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = 0 \quad \text{or} \quad \frac{\mathbf{v}_1}{\mathbf{v}_2} = -\frac{m_2}{m_1}$$

$$\frac{\mathbf{v}_1 + \mathbf{v}_2}{\mathbf{v}_1} = \frac{m_2 - m_1}{m_2} \quad \text{or} \quad (\mathbf{v}_1 + \mathbf{v}_2) = \frac{(m_2 - m_1)\mathbf{v}_1}{m_2}$$

For the kinetic energy T, we have

$$\frac{1}{2}m_1 \mathrm{v}_1^2 + \frac{1}{2}m_2 \mathrm{v}_2 = T$$

Replacing $\mathbf{v}_2$ from the above and simplifying

$$\mathrm{v}_1^2 = \frac{2T}{m_1}\frac{m_2}{m_1 + m_2} \quad \text{and} \quad \mathrm{v}_2^2 = \frac{2T}{m_2}\frac{m_1}{m_1 + m_2}$$

$$\mathrm{v}_1^2 - \mathrm{v}_2^2 = \frac{2T(m_2 - m_1)}{m_1 m_2} \quad \text{or} \quad (\mathrm{v}_1 + \mathrm{v}_2)(\mathrm{v}_1 - \mathrm{v}_2) = \frac{2T\,(m_2 - m_1)}{m_1 m_2}$$

Substituting the value of $\mathrm{v}_1 + \mathrm{v}_2$ and simplifying

$$\mathrm{v}_1 - \mathrm{v}_2 = \frac{2T}{m_1 \mathbf{v}_1} = \frac{2T}{m_1}\sqrt{\frac{m_1}{2T}\frac{(m_1 + m_2)}{m_2}}$$

$$= \sqrt{\frac{2Tm}{m_1 m_2}}$$

which is the relative velocity of one with respect to the other.

Example 2.2 Ball A of mass m is attached to one end of a rigid massless rod of length $2l$, while an identical ball B is attached to the centre of the rod. This arrangement is held by the empty end and is whirled around in a horizontal circle at a constant rate. Ball A travels at a constant speed of v_A. Find the tension in each half of the rod.

Solution: Fig. 2.4 illustrates the details.

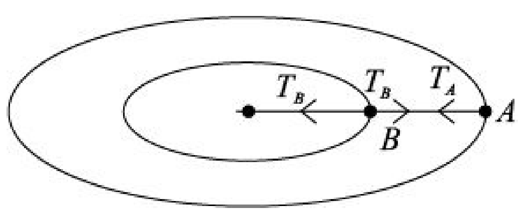


Fig. 2.4 Illustration showing balls A and B of example 2.2.

Ball A: Only a single tension force T_A acts on A. This provides the centripetal force keeping ball A on its circular path.

$$T_A = \frac{m\mathrm{v}_A^2}{2l}$$

Ball B: Two tension forces act on ball B. The centripetal force is provided by $T_B - T_A$.

$$T_B - T_A = \frac{m\mathrm{v}_B^2}{l}$$

Since the arrangement is a rigid unit, the time for each ball to travel once around is the same. Since the radius of the circular path of B is half that of A, $\mathbf{v}_B = \mathbf{v}_A/2$. From the second equation

$$T_B = \frac{m\mathrm{v}_B^2}{l} + T_A = \frac{m\mathrm{v}_A^2}{4l} + \frac{m\mathrm{v}_A^2}{2l}$$

$$T_B = \frac{m\mathrm{v}_A^2}{2l}\left(\frac{1}{2} + 1\right) = \frac{3m\mathrm{v}_A^2}{4l}$$

Example 2.3 The maximum possible exhaust velocity of a rocket is 2 km/s. Calculate the mass ratio for the rocket if it is to attain the escape velocity of 11.2 km/s. Also calculate the time taken by the rocket to attain this velocity if its rate of change of mass to its initial mass is 1/10.

Solution: The velocity $\mathrm{v} = u\, ln \frac{m_0}{m}$ or $ln \frac{m_0}{m} = \frac{\mathrm{v}}{u}$

$$ln \frac{m_0}{m} = \frac{\mathrm{v}}{u} = \frac{11.2\,\text{km/s}}{2\,\text{km/s}} = 5.6$$

$$\frac{m_0}{m} = 270.46 \cong 270$$

From Eq. (2.40) we have $m = m_0 - \alpha t$. It is given that $\frac{\alpha}{m_0} = \frac{1}{10}$. Substituting this value of α, we have

$$m = m_0 - \frac{m_0}{10}t \quad \text{or} \quad \frac{m}{m_0} = \left(1 - \frac{t}{10}\right)$$

$$t = 10\left(1 - \frac{m}{m_0}\right) = 10\left(1 - \frac{1}{270}\right) = 9.96 \text{ s}$$

Example 2.4 Masses of 1, 2 and 3 kg are located at positions $4\hat{j} + 3\hat{k}$, $4\hat{j} + 3\hat{k}$, and $2\hat{i} + 2\hat{k}$ respectively. If their velocities are $7\hat{i}, -6\hat{j}$ and $-3\hat{i}$, find the position and velocity of the centre of mass. Also, find the angular momentum of the system with respect to the origin.

Solution: Radius vector of the centre of mass

$$\mathbf{R} = \sum_i \frac{m_i \mathbf{r}_i}{M} = \frac{1(\hat{i} + \hat{j} + \hat{k}) + 2(4\hat{j} + 3\hat{k}) + 3(2\hat{i} + 2\hat{k})}{6}$$

$$= \frac{(7\hat{i} + 9\hat{j} + 13\hat{k})}{6}$$

Velocity of the centre of mass

$$\mathbf{V} = \frac{\sum_i m_i \mathbf{v}_i}{M} = \frac{1 \times 7\hat{i} + 2(-6\hat{j}) + 3(-3\hat{i})}{6}$$

$$= \frac{-2\hat{i} - 12\hat{j}}{6} = \frac{-\hat{i} - 6\hat{j}}{3}$$

The angular momentum vector about the origin

$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i$$

$$= (\hat{i} + \hat{j} + \hat{k}) \times 7\hat{i} + (4\hat{j} + 3\hat{k}) \times 2(-6\hat{j}) + (2\hat{i} + 2\hat{k}) \times 3(-3\hat{i})$$

$$= 7\hat{j} - 7\hat{k} + 36\hat{i} - 18\hat{j} = 36\hat{i} - 11\hat{j} - 7\hat{k}$$

Example 2.5 Particles of masses 1, 2 and 4 kg move under a force such that their position vectors at time *t* are respectively $\mathbf{r}_1 = 2\hat{i} + 4t^2\hat{k}, \mathbf{r}_2 = 4t\,\hat{i} - \hat{k},$ $\mathbf{r}_3 = (\cos \pi t)\hat{i} + (\sin \pi t)\,\hat{j}$. Find the angular momentum of the system about the origin at t = 1 s.

Solution: The angular momentum **L** is given by

$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i$$

$$= (2\hat{i} + 4t^2\hat{k}) \times 8t\,\hat{k} + (4t\,\hat{i} - \hat{k}) \times 8\hat{i} + [(\cos \pi t)\,\hat{i} + (\sin \pi t)\,\hat{j}]$$

$$\times 4\pi\,[(-\sin \pi t)\hat{i} + (\cos \pi t)\,\hat{j}]$$

$$= -16t\,\hat{j} - 8\,j + 4\pi\,(\cos^2 \pi t + \sin^2 \pi t)\,\hat{k}$$

$$(\mathbf{L})_{t\,=\,1s} = -\,24\,\hat{j} + 4\pi\,\hat{k}$$

REVIEW QUESTIONS

1. Define the centre of mass of a system of particles. What is the centre of mass frame of reference?
2. Is mass necessary at the centre of mass in the case of a solid body? Explain.
3. In a system of particles, if Newton's third law is applicable for the internal forces, show that the acceleration of the centre of mass is only due to the external forces.
4. Show that the centre of mass of a system of particles moves as if the total external force were acting in the entire mass of the system concentrated at the centre of mass.
5. Explain the principle of a rocket. What is meant by thrust of a rocket? On what factors does thrust depend?
6. The final velocity of a multistage rocket is much greater than the final velocity of a single-stage rocket of the same total weight and fuel supply. Explain.
7. What is a closed system? For a closed system, show that the linear momentum is conserved.
8. The total angular momentum of a system of particles about a point is equal to the sum of angular momentum of the system concentrated at the centre of mass and the angular momentum of the system of particles about the centre of mass. Substantiate.

PROBLEMS

1. A rocket motor consumes 120 kg of fuel per second. If the exhaust velocity is 5 km/s, what is the thrust on the rocket? What would be the velocity of the rocket when its mass is reduced to 1/15th of its initial mass? Assume that the initial velocity of the rocket is zero.
2. Calculate the mass ratio (m_0/m) of a rocket so that its speed is: (i) equal to the exhaust speed; (ii) equal to twice the exhaust speed. Here m_0 is the initial mass and m is the mass, at a time, of the rocket.
3. In a system of particles, the force exerted by the ith particle on the jth one is $\mathbf{F}_{ij}$. If Newton's third law is applicable for the internal forces, show that $\sum_i \sum_j \mathbf{r_i} \times \mathbf{F}_{ij} = 0, \quad i \neq j.$
4. In a system of particles, if Newton's third law is applicable for the internal forces, show that $\sum_i \sum_j \mathbf{F}_{ij} \cdot d\mathbf{r}_i = \frac{1}{2} \sum_i \sum_j \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \quad i \neq j$

where $\mathbf{F}_{ij}$ is the force exerted by the ith particle on the jth one and $d\mathbf{r}_{ij} = d\mathbf{r}_i - d\mathbf{r}_j$.

5. A string with masses m_1 and m_2 at its ends passes over a smooth pulley fixed at the edge of a table, with the mass m_1 resting on the smooth table and m_2 hanging. If $m_2 > m_1$, calculate the acceleration of the masses and the tension in the string.
6. Particles of masses 4, 3 and 1 kg move under a force such that their position vectors at time t are $\mathbf{r}_1 = 3\hat{j} + 2t^2\hat{k}$, $\mathbf{r}_2 = 3t\hat{i} - \hat{k}$ and $\mathbf{r}_3 = 4t\hat{i} + t^2\hat{j}$. Find the position of the centre of mass and the angular momentum of the system about the origin at $t = 2$ s.
7. The position vectors and velocity of masses 2 kg, 3 kg and 4 kg are respectively $2\hat{i} - 3\hat{j}$, $\hat{i} + \hat{j} + \hat{k}$ and $4\hat{j} + 3\hat{k}$. If their velocities are $-3\hat{i}$, $-6\hat{j}$ and $2\hat{i} + 3\hat{k}$ units respectively, find the position and velocity of the centre of mass. Also evaluate the total angular momentum vector of the system with respect to the origin.
8. In a radioactive decay of a nucleus, an electron and a neutrino are emitted at right angles to each other. Their momenta are 1.3 10^{-22} and 6.0 10^{-29}

kgm/s. If the mass of the residual nucleus is 6.0 10^{-26} kg, calculate the recoil kinetic energy.

3

Lagrangian Formulation

In the previous chapters we were able to demonstrate the effectiveness of Newton's laws of motion in solving variety of problems. However, if the system is subject to external constraints, solving the equations of motion may be difficult, and sometimes it may be difficult even to formulate them. The forces of constraints are usually very complex or unknown, which makes the formalism more difficult. To circumvent these difficulties, two different methods, Lagrange's and Hamilton's formulations, have been developed. These techniques use an energy approach and are constructed in such a way that the Newtonian formalism follows from it. Before going over to these procedures, we try to understand certain terms such as constraints, generalized coordinates, *etc*. In this chapter a discussion on the Lagrangian formalism is given.

3.1 CONSTRAINTS

A motion that cannot proceed arbitrarily in any manner is called a **constrained motion**. The conditions which restricts the motion of the system are called **constraints**. For example, gas molecules within a container are constrained by the walls of the vessel to move only inside the container. A particle placed on the surface of a solid sphere is restricted by the constraint, so that it can only move on the surface or in the region exterior to the sphere. There are two main types of constraints, holonomic and non-holonomic.

Holonomic Constraints

In holonomic constraints, the conditions of constraint are expressible as equations connecting the coordinates and time, having the form $f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots \mathbf{r}_n, t) = 0$ (3.1) We give below a few typical examples of holonomic constraint: (i) In a rigid body, the distance between any two particles of the body remains constant during motion. This is expressible as $|\mathbf{r}_i - \mathbf{r}_j|^2 = c_{ij}^2$ (3.2) where c_{ij} is the distance between the particles *i* and *j* at $\mathbf{r}_i$ and $\mathbf{r}_j$.

(ii) The sliding of a bead on a circular wire of radius *a* in the *xy*-plane is another example. The equation of constraint is $x^2 + y^2 = a^2$ (3.3) which can also be expressed in the differential form as $x\,dx + y\,dy = 0$ (3.3a) Equations (3.2) and (3.3) are of the same form as Eq. (3.1). The differential equation denoted by Eq. (3.3a) can be integrated to obtain Eq. (3.3). Holonomic constraints are also known as **integrable constraints**. The term *integrable* is used here since Eq. (3.1) is equivalent to the differential equation $\sum_i \frac{\partial f}{\partial r_i} d\mathbf{r}_i = 0$ (3.4) Equation (3.4) can be readily integrated to Eq. (3.1).

Non-holonomic Constraints

Non-holonomic constraints are those which are not expressible in the form of Eq. (3.1). The coordinates in this case are restricted either by inequalities or by non-integrable differentials.

(i) The constraint involved in the example of a particle placed on the surface of a sphere is non-holonomic, which may be expressed as the inequality $r^2 - a^2 \geq 0$ (3.5) where *a* is the radius of the sphere.

(ii) Gas molecules in a spherical container of radius *R*. If $\mathbf{r}_i$ is the position vector of the *i*th molecule, $x_i^2 + y_i^2 + z_i^2 \leq R^2$ (3.6) Here, the centre of the sphere is the origin of the coordinate system.

In non-holonomic constraints, if the constraints are expressible as relations among the velocities of the particles of the system, that is, $f(x_1, x_2, \ldots, \dot{x}_1, \dot{x}_2, \ldots t) = 0$ (3.6a) and if these equations of non-holonomic constraints can be integrated to give relations among the coordinates, then the constraints become holonomic.

Scleronomous and Rheonomous Constraints Constraints are further classified as scleronomous and rheonomous.

A scleronomous constraint is one that is independent of time

whereas a rheonomous constraint contains time explicity. A pendulum with an inextensible string of length l_0 is described by the equation $x^2 + y^2 = l_0^2$ **(3.7) As the constraint equation is independent of time, it is a scleronomous constraint. A pendulum with an extensible string is rheonomous, the condition of constraint is** $x^2 + y^2 = l^2(t)$ **(3.8) where $l(t)$ is the length of the string at time t.**

Constraints introduce two types of difficulties in the solution of mechanical problems. The coordinates r_i are no longer independent as they are connected by the equations of constraint. In the case of holonomic constraints, this difficulty is solved by the introduction of generalized coordinates. The second difficulty is due to the fact that the forces of constraints cannot be specified explicitly. They are among the unknowns of the problem and must be obtained from the solution. This difficulty can be solved if the problem is formulated in the Lagrangian form, in which the forces of constraint do not appear.

In most of the systems of interest, the constraints involved are holonomic. Hence, we restrict ourselves mainly to holonomic systems.

3.2 GENERALIZED COORDINATES

Degrees of Freedom

The number of independent ways in which a mechanical system can move without violating any constraint is called the **number of degrees of freedom** of the system. It is the minimum possible number of coordinates required to describe the system completely. When a particle moves in space, it has three degrees of freedom. If it is constrained to move along a space curve it has only one degree of freedom whereas it has two degrees of freedom if it moves in a plane.

Generalized Coordinates

For a system of N particles, free from constraints, we require a total of $3N$ independent coordinates to describe its configuration completely. Let there are k

constraints of the type $f_s(\mathbf{r}_1, \mathbf{r}_2,..., \mathbf{r}_N,t) = 0 \quad s = 1, 2, 3,..., k$ (3.9) acting on the system. Now the system has only $3N - k$ independent coordinates or degrees of freedom. These $3N - k$ independent coordinates represented by the variables $q_1, q_2, q_3,..., q_{3N-k}$ are called the **generalized coordinates**. In terms of the new coordinates, the old coordinates $r_1, r_2,..., r_N$ can be written as

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2,..., q_{3N-k}, t)$$
$$\mathbf{r}_2 = \mathbf{r}_2(q_1, q_2,..., q_{3N-k}, t) \qquad (3.10)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathbf{r}_N = \mathbf{r}_N(q_1, q_2,..., q_{3N-k}, t)$$

These are the transformation equations from the set of r_l variables to q_l variables. In analogy with cartesian coordinates, time derivatives $\dot{q}_1, \dot{q}_2, \dot{q}_3...$ are defined as **generalized velocities**.

Generalized coordinates are not unique. They may or may not have dimensions of length. Depending on the problem, it may prove more convenient to select some of the coordinates with dimensions of energy, some others with dimensions of L^2, and yet some others could be combinations of angles and coordinates, and so on.

Configuration Space

We have seen that the configuration of a system can be specified completely by the values of $n = 3N - k$ independent generalized coordinates $q_1, q_2,..., q_n$. It is convenient to think of the n q's as the coordinates of a point in an n-dimensional space. This n-dimensional space is called the **configuration space** with each dimension represented by a coordinate. As the generalized coordinates are not necessarily position coordinates, configuration space is not necessarily connected to the physical 3-dimensional space and the path of motion also does not necessarily resemble the path in space of actual particle.

3.3 PRINCIPLE OF VIRTUAL WORK

A **virtual displacement**, denoted by $d\mathbf{r}_i$, refers to an imagined, infinitesimal, instantaneous displacement of the coordinate that is consistent with the constraints. It is different from an actual displacement $d\mathbf{r}_i$ of the system occurring in a time interval *dt*. It is called *virtual* as the displacement is instantaneous. As there is no actual motion of the system, the work done by the forces of constraint in such a virtual displacement is zero.

Consider a scleronomic system of *N* particles in equilibrium. Let $\mathbf{F}_i$ be the force acting on the *i*th particle. The force $\mathbf{F}_i$ is a vector addition of the externally applied force $\mathbf{F}_i^e$ and the forces of constraints $\mathbf{f}_i$. Then $\mathbf{F}_i = \mathbf{F}_i^e + \mathbf{f}_i$ (3.11)

If $d\mathbf{r}_i$ is a virtual displacement of the *i*th particle, the virtual work done dW_i on the *i*th particle is given by

$\delta W_i = \mathbf{F}_i . \delta\mathbf{r}_i$ (3.12) If the system is in equilibrium, the total force on each particle must be zero:

$\mathbf{F}_i = 0$ for all *i*. Therefore, the dot product $\mathbf{F}_i . \delta\mathbf{r}_i$ is also zero. That is,

$\delta W_i = (\mathbf{F}_i^e + \mathbf{f}_i) . \delta\mathbf{r}_i = 0 \qquad i = 1, 2, \ldots, N$ (3.13) The total virtual work done on the system d*W* is the sum of the above vanishing products:

$$\delta W = \sum_{i=1}^{N} \delta W_i = \sum_{i=1}^{N} (\mathbf{F}_i^e + \mathbf{f}_i) . \delta\mathbf{r}_i = 0$$

$$= \sum_{i=1}^{N} F_i^e . \delta\mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i . \delta\mathbf{r}_i = 0 \qquad (3.14)$$

Under a virtual displacement, the work done by the forces of constraints is zero. This is valid for rigid bodies and most of the constraints that commonly occur. Therefore, Eq. (3.14) reduces to

$$\delta W = \sum_{i=1}^{N} \boldsymbol{F}_i^e . \delta \boldsymbol{r}_i = 0 \qquad (3.15)$$

which is the **principle of virtual work** and is stated as : *In an N-particle system, the total work done by the external forces when virtual displacements are made is called virtual work and the total virtual work done is zero*.

The coefficients $\delta\mathbf{r}_i$ in Eq. (3.15) can no longer be set equal to zero as they are not independent. It should also be noted that the principle of virtual work deals only with statics.

3.4 D'ALEMBERT'S PRINCIPLE

The principle of virtual work deals only with statics and the general motion of the system is not relevant here. A principle that involves the general motion of the system was suggested by D' Alembert.

Consider the motion of an N-particle system. Let the force acting on the *i*th particle be $\mathbf{F}_i$. By Newton's law $\mathbf{F}_i = \mathbf{p}_i$ or $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$ (3.16) This means that the *i*th particle in the system will be in equilibrium under a force equal to the actual force plus a "reversed effective force", $-\dot{\mathbf{p}}_i$, as named by D'Alembert. Then dynamics reduces to statics. To this equivalent static problem, give a virtual displacement $\delta\mathbf{r}_i$ which leads to

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i).\delta\mathbf{r}_i = 0 \qquad (3.17)$$

$$\sum_{i=1}^{N} (\mathbf{F}_i^e + \mathbf{f}_i - \dot{\mathbf{p}}_i).\delta\mathbf{r}_i = 0 \qquad (3.18)$$

Restricting to situations where the virtual work done by forces of constraints is zero $\sum_{i=1}^{N} (\mathbf{F}_i^e - \dot{\mathbf{p}}_i).\delta\mathbf{r}_i = 0$ (3.19) which is **D'Alembert's principle**.

3.5 LAGRANGE'S EQUATIONS

Lagrange used D'Alembert's principle as the starting point to derive the equations of motion, now known as Lagrange's equations. Dropping the superscript *e* in Eq. (3.19) $\sum_{i=1}^{N} (F_i - \dot{\mathbf{p}}_i).\delta\mathbf{r}_i = 0$ (3.20) The virtual displacements $\delta\mathbf{r}_i$ in Eq. (3.20) are not independent. Lagrange changed Eq. (3.20) into an equation involving virtual displacment of the generalized coordinates which are independent.

Consider a system with N particles at $\mathbf{r}_1$, $\mathbf{r}_2$,..., $\mathbf{r}_N$ having k equations of holonomic constraints. The system will have $n = 3N - k$ generalized coordinates q_1, q_2,...,q_n. The transformation equations from the r variables to the q variables are given by Eq. (3.10).

$\mathbf{r}_i = \mathbf{r}_i (q_1, q_2, ..., q_n, t)$ (3.21) Since virtual displacement does not involve time, from Eq. (3.21)

$$\delta\mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_n}\delta q_n = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \qquad (3.22)$$

Here dq_j's are the virtual displacements of generalized coordinates. From Eq. (3.21) we also have $\dot{\mathbf{r}}_i = \frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \mathbf{r}_i}{\partial t}$

$$= \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \qquad (3.23)$$

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \qquad (3.24)$$

The form of D'Alembert's principle, Eq. (3.20), can be changed easily by substituting δr_i from Eq. (3.22). The first term of Eq. (3.20) is

$$\sum_i \mathbf{F}_i . \delta\mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \right) = \sum_j \left(\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right)\delta q_j$$

$$= \sum_{j=1}^{n} Q_j \delta q_j \qquad (3.25)$$

where,

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \qquad (3.26)$$

The quantity Q_j is the jth component of the generalized force **Q**. The generalized force components need not have the dimension of force as the q's need not have the dimension of length. However, Q_jdq_j must have the dimension of work.

We now write the inertial force term of Eq. (3.20)

$$\sum_i \dot{\mathbf{p}}_i . \delta\mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i . \delta\mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_j \left[\sum_i \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r_i}}{\partial q_j} \right) - \sum_i m_i \dot{\mathbf{r}}_i \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j \tag{3.27}$$

Using Eq. (3.24)

$$\sum_i \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \sum_i \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_\mathbf{i}}{\partial \dot{q}_j} \right) = \frac{d}{dt} \left(\sum_i m_i \mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right)$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \right) = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} \tag{3.28}$$

where T is the total kinetic energy of the system. Changing the order of differentiation in the second term of Eq. (3.27) $\sum_i m_i \dot{\mathbf{r}}_i \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i m_i \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \right) = \frac{\partial T}{\partial q_j}$ (3.29) Use of Eqs. (3.28) and (3.29) reduces Eq. (3.27) to $\sum_i \dot{\mathbf{p}}_i . \delta\mathbf{r}_i = \sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right)$ (3.30) With Eqs. (3.26) and (3.30), Eq. (3.20) becomes $\sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$ (3.31) The *dq's* are independent and therefore each of the coefficients must separately vanish. From which it follows that $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0 \quad j = 1, 2, ..., n$ (3.32) Equation (3.32) can be simplified further if the external forces $\mathbf{F}_i$ are conservative: $\mathbf{F}_i = -\nabla_i V$ where $V = V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$. Then

$$Q_j = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial \mathbf{r}_i} \frac{\partial \mathbf{r}_i}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j} \tag{3.33}$$

since

$$dV = \sum_i \frac{\partial V}{\partial \mathbf{r}_i} d\mathbf{r}_i = \sum_i \frac{\partial V}{\partial \mathbf{r}_i} \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j \right)$$

$$\frac{\partial V}{\partial q_j} = \sum_i \frac{\partial V}{\partial \mathbf{r}_i} \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{3.34}$$

Equation (3.32) becomes

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \quad j = 1, 2, \ldots, n \tag{3.35}$$

If the potential V is a function of position only, $\left(\frac{\partial V}{\partial \dot{q}_j} \right) = 0.$ We can now include this term in Eq. (3.35). Then

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} (T - V) - \frac{\partial}{\partial q_j} (T - V) = 0 \quad j = 1, 2, \ldots, n \tag{3.36}$$

We now introduce a new function L defined by $L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q)$ (3.37) where q stands for q_1, q_2, q_3,..., q_n and $\dot{q}$ stands for $\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n$. This function L is called the **Lagrangian function** of the system. In terms of L, Eq. (3.36) becomes $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad j = 1, 2, 3, \ldots, n$

(3.38) These n equations, one for each independent generalized coordinate, are known as **Lagrange's equations**. These constitute a set of n second order differential equations for n unknown functions $q_j(t)$ and the general solution contains
$2n$ integration constants.

In certain systems the forces acting are not conservative, say where a part is derivable from a potential and the other is dissipative. In such cases, Lagrange's equations can be written as $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q'_j \quad j = 1, 2, 3, \ldots, n$ (3.39)

where L contains the potential of the conservative forces and Q'_j represents the force not arising from that potential.

3.6 KINETIC ENERGY IN GENERALIZED COORDINATES

Kinetic energy of a particle of mass m is a homogeneous quadratic function of the velocities $\dot{\mathbf{r}}_i$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2}\sum_i m_i\,(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \tag{3.40}$$

Replacing $\dot{\mathbf{r}}_i$ using Eq. (3.23),

$$T = \frac{1}{2}\sum_i m_i \left[\sum_{j=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}\right] \cdot \left[\sum_{k=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}\right]$$

we have

$$= \frac{1}{2}\sum_i \sum_j \sum_k m_i \frac{\partial \mathbf{r}_i}{\partial q_j}\frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_j\,\dot{q}_k$$

$$+ \frac{1}{2}\sum_i m_i \left(2\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial t}\dot{q}_j\right) + \frac{1}{2}\sum_i m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial t}\right) \tag{3.41}$$

$$T(q, \dot{q}, t) = \sum_j \sum_k a_{jk}\dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c \tag{3.42}$$

where,

$$a_{jk} = \frac{1}{2}\sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_j}\frac{\partial \mathbf{r}_i}{\partial q_k} \quad b_j = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_j}\frac{\partial \mathbf{r}_i}{\partial t} \quad c = \frac{1}{2}\sum_i m_i \left(\frac{\partial \mathbf{r}_i}{\partial t}\right)^2 \tag{3.43}$$

In general, the kinetic energy in terms of generalized coordinates consists of three distinct terms: the first term contains quadratic terms, the second contains linear terms and the third is independent of generalized velocities. If

$$\frac{\partial \mathbf{r}_i}{\partial q_j}\frac{\partial \mathbf{r}_i}{\partial q_k}=0 \quad \text{for } j \neq k \qquad (3.44)$$

the generalized coordinate system in the q_j's is referred to as an **orthogonal system**.

The special case where time does not appear explicitly in the transformation equations, $(\partial \mathbf{r}_i/\partial t)=0$ and therefore b_j = c = 0, and Eq. (3.42) reduces to

$$T=\sum_j \sum_k a_{jk}\,\dot{q}_j\,\dot{q}_k \qquad (3.45)$$

That is, the kinetic energy is a homogeneous quadratic function of the generalized velocities. In such a case, we are led to an interesting result when T is differentiated with respect to $\dot{\mathbf{q}}_l$. For this, let us go back to the first term of Eq. (3.41)

$$\frac{\partial T}{\partial \dot{q}_l}=\frac{1}{2}\sum_i m_i\left[\frac{\partial \mathbf{r}_i}{\partial q_l}\left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k\right)+\left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j\right)\frac{\partial \mathbf{r}_i}{\partial q_l}\right]$$

$$=\sum_i m_i\left(\frac{\partial \mathbf{r}_i}{\partial q_l}\right)\left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k\right) \qquad (3.46)$$

Multiplying both sides by $\dot{q}_l$ and summing over l

$$\sum_{l=1}^{n}\dot{q}_l\frac{\partial T}{\partial \dot{q}_l}=\sum_i m_i\left(\sum_l \frac{\partial \mathbf{r}_i}{\partial q_l}\dot{q}_l\right)\left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k\right)=2T \qquad (3.47)$$

This result is a special case of Euler's theorem, which states that if f is a homogeneous function of the y_k, which is of degree n, then

$$\sum_k y_k\frac{\partial f}{\partial y_k}=nf \qquad (3.48)$$

In the present case, T is a homogeneous quadratic function of the generalized velocities $\dot{q}_i$. Hence, from Eq. (3.48) we have

$$\sum_k \dot{q}_k\frac{\partial T}{\partial \dot{q}_k}=2T \qquad (3.49)$$

which is Eq. (3.47).

3.7 GENERALIZED MOMENTUM

Consider the motion of a particle of mass m moving along x-axis. Its linear momentum **p** is $m\dot{x}$ and kinetic energy $T = (1/2)\ m\ \dot{x}^2$. Differentiating T with respect to $\dot{x}$ we have $\frac{\partial T}{\partial \dot{x}} = m\dot{x} = \mathbf{p}$ (3.50) If the potential V is not a function of the velocity $\dot{x}$, since $L = T - V$

$\mathbf{p} = \frac{\partial T}{\partial \dot{x}} = \frac{\partial L}{\partial \dot{x}}$ (3.51) Let us use this concept to define generalized momentum. For a system described by a set of generalized coordinates q_1, q_2,..., q_n, we define **generalized momentum** $\mathbf{p}_i$ corresponding to generalized coordinate q_i as $\mathbf{p}_i = \frac{\partial L}{\partial \dot{q}_i}$ (3.52) Sometimes it is also known as **conjugate momentum** (conjugate to co-ordinate q_i).

In general, generalized momentum is a function of the q's, $\dot{q}$'s and t . As the Lagrangian is utmost quadratic in the $\dot{q}$'s, $\mathbf{p}_i$ is a linear function of the $\dot{q}$'s. The generalized momentum $\mathbf{p}_i$ need not always have the dimension of linear momentum. However, the product of any generalized momentum and the associated coordinate must always have the dimension of angular momentum. For a conservative system, the use of the expression for generalized momentum, Eq. (3.52), reduces Lagrange's equations of motion to $\dot{\mathbf{p}}_j = \frac{\partial L}{\partial q_j} \quad j = 1, 2, \ldots, n$

(3.53)

3.8 FIRST INTEGRALS OF MOTION AND CYCLIC

COORDINATES

Lagrange's equations of motion for a system having n degrees of freedom will have n differential equations that are second order in time. As the solution of each equation requires two integration constants, a total of $2n$ constants have to be evaluated from the initial values of n-generalized coordinates and n-generalized velocities. In general, it is either very difficult to solve the problem completely or very tedious. However, very often a great deal of information about the system is possible from the first integrals of equations of motion. The *first integrals of motion* are functions of the generalized coordinates q's and generalized velocities $\dot{q}$'s of the form

$$f(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) = \alpha_i \text{ (constant)} \qquad (3.54)$$

These first integrals are of interest because they give good deal of information about the system. The conservation laws of energy, momentum and angular momentum that we deduced in Newtonian formalism are of this type. In the process, the relation between conservation laws and the symmetry properties of the system is revealed.

Cyclic Coordinates

Coordinates that do not appear explicitly in the Lagrangian of a system (although it may contain the corresponding generalized velocities) are said to be **cyclic or ignorable**. If q_i is a cyclic coordinate

$$L = L(q_1, q_2, ...q_{i-1}, q_{i+1}, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) \qquad (3.55)$$

In such a case $(\partial L/\partial q_i) = 0$ and Lagrange's equation reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial \dot{q}_i} = \text{constant } \alpha_i$$

which means that

$$\frac{\partial L}{\partial \dot{q}_i} = \mathbf{p}_i = \text{constant } \alpha_i \qquad (3.56)$$

Equation (3.56) constitutes a first integral for the equations of motion. We may state this result as a general conservation theorem: *The generalized momentum conjugate to a cyclic coordinate is conserved during the motion.*

3.9 CONSERVATION LAWS AND SYMMETRY PROPERTIES

The title suggests the possibility of a relationship between the conservation laws and symmetries. In this section, we shall investigate the connection between the two in detail. A closed system is one that does not interact with other systems.

Homogeneity of Space and Conservation of Linear Momentum Homogeneity in space means that the mechanical properties of a closed system remain unchanged by any parallel displacement of the entire system in space. That means that the Lagrangian is unchanged (*dL* = 0) if the system is displaced by an infinitesimal amount $\delta \mathbf{r} : \mathbf{r}_i \to \mathbf{r}_i + \delta \mathbf{r}_i$. **The change in *L* due to infinitesimal displacement *d*r, the velocities remaining fixed, is given by**

$$\delta L = \sum_i \frac{\partial L}{\partial \mathbf{r}_i} \delta \mathbf{r}_i$$

(3.57) The second term in this equation vanished as velocities remained constant $(\delta \dot{\mathbf{r}}_i = 0)$. **Since each of the** $\delta \mathbf{r}_i$ **in Eq. (3.57) is an arbitrary independent displacement, the coefficient of each term is zero separately. Hence,** $\frac{\partial L}{\partial \mathbf{r}_i} = 0$

(3.58) With this condition, Lagrange's equation reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \text{constant}$$

$$\mathbf{p}_i = \text{constant}$$

(3.59) As the $\mathbf{p}_i$'s are additive, the total linear momentum **p** of a closed system is a constant. Thus, the homogeneity of space implies that the linear momentum **p** is a constant of motion.

It can also be proved that if the Lagrangian of a system (not necessarily closed) is invariant with respect to translation in a certain direction, then the linear momentum of the system in that direction is constant in time.

Isotropy of Space and Conservation of Angular Momentum Space is isotropic, which means the mechanics (i.e., the Lagrangian) of a closed system is unaffected by an infinitesimal rotation of the system in space, i.e.,
***dL* = 0. Consider a cartesian frame of reference with O as the origin. Let $\mathbf{r}_i$ be the radius vector of the *i*th particle located at *P*. Let the system as a whole undergoes an infinitesimal rotation *df*. The displacement is denoted by the vector *df* and its direction is that of the axis of rotation. Due to this rotation, the position vector**

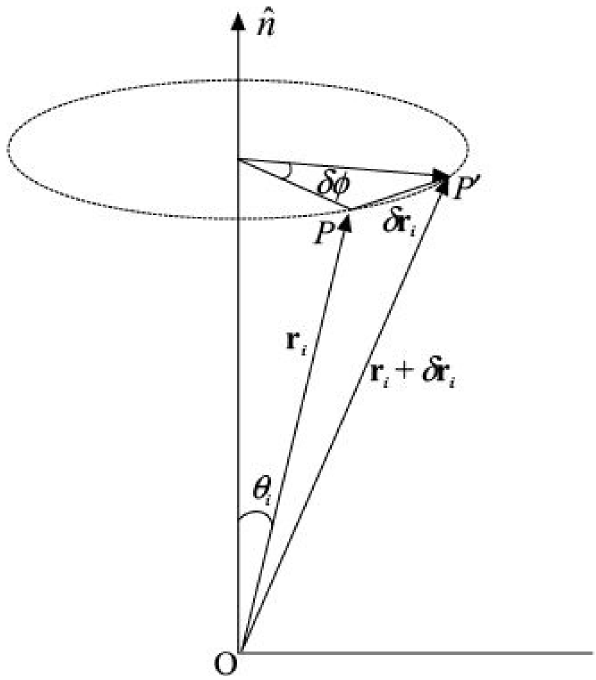
of the ith particle is shifted from *P* to *P* and the radius vector $\mathbf{r}_i$ to $\mathbf{r}_i + d\mathbf{r}_i$

(see Fig. 3.1)

$$|\delta \mathbf{r}_i| = \mathbf{r}_i \sin\theta_i \; \delta\phi$$

$$\delta \mathbf{r}_i = \delta\boldsymbol{\phi} \times \mathbf{r}_i \tag{3.60}$$

When the system is rotated, the position vectors of all particles change their directions in this way. The corresponding change in the velocity vector is given



by $\delta \dot{\mathbf{r}}_i = \delta \mathbf{f} \times \dot{\mathbf{r}}_i$ (3.61)

Fig. 3.1 Change of a position vector under rotation of the system.

The condition that $dL = 0$ leads to

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \mathrm{r}_i} \delta \mathbf{r}_i + \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \delta \dot{\mathbf{r}}_i \right) = 0 \tag{3.62}$$

Equations (3.38) and (3.52) give

$$\frac{\partial L}{\partial \mathbf{r}_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \dot{\mathbf{p}}_i \quad \text{and} \quad \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \mathbf{p}_i$$

Equations (3.62) now becomes

$$\sum_i (\dot{\mathbf{p}}_i \, \delta \mathbf{r}_i + \mathbf{p}_i \, \delta \dot{\mathbf{r}}_i) = 0 \tag{3.63}$$

Substituting $\delta \mathbf{r}_i$ from Eq.(3.60) and $\delta \dot{\mathbf{r}}_i$ from Eq. (3.61)

$$\sum_i [\dot{\mathbf{p}}_i \, (\delta \boldsymbol{\phi} \times \mathbf{r}_i) + \mathbf{p}_i \, (\delta \boldsymbol{\phi} \times \dot{\mathbf{r}}_i)] = 0$$

Using the vector relation **A.(B × C) = B.(C × A) = C.(A × B)**, the above relation can be written as

$$\sum_i [\delta \boldsymbol{\phi}.(\mathbf{r}_i \times \dot{\mathbf{p}}_i) + \delta \boldsymbol{\phi}.(\dot{\mathbf{r}}_i \times \mathbf{p}_i)] = 0$$

$$\delta \boldsymbol{\phi}. \sum_i \frac{d}{dt} (\mathbf{r}_i \times \mathbf{p}_i) = 0$$

$$\delta \boldsymbol{\phi} \cdot \frac{d\mathbf{L}}{dt} = 0 \tag{3.64}$$

where **L** is the total angular momentum of the system. Since *df* is arbitrary

$$\frac{d\mathbf{L}}{dt} = 0$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \text{constant} \tag{3.65}$$

Thus, the rotational invariance of the Lagrangian of a closed sytem is equivalent to the conservation of total angular momentum.

Homogeneity of Time and Conservation of Energy Homogeneity in time implies that the Lagrangian of a closed system does not depend explicitly on the time *t*. That is, $(\partial L / \partial t) = 0$. The total time

derivative of the Lagrangian is

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} \tag{3.66}$$

Use of the condition that $(\partial L/\partial t) = 0$ gives

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{3.67}$$

Replacing $(\partial L/\partial q_i)$ using Lagrange's equation, we have

$$\frac{dL}{dt} = \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \sum_i \frac{d}{dt}(p_i\dot{q}_i)$$

$$\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - L\right) = 0 \tag{3.68}$$

That is, the quantity in parenthesis must be constant in time. Denoting the constant by H called the **Hamiltonian** of the system $\sum_i p_i\dot{q}_i - L = H$ (constant) (3.69) It can be shown that H is the total energy of the system if (i) the potential energy V is velocity-independent and (ii) the transformation equations connecting the rectangular and generalized coordinates do not depend on time explicitly. When condition (ii) is satisfied, the kinetic energy T is a homogeneous quadratic function of the generalized velocities and by Euler's theorem, Eq. (3.48) $\sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = 2T$ (3.70) Now, Eq. (3.69) can be written as

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i \frac{\partial (T-V)}{\partial \dot{q}_i}\dot{q}_i - L$$

$$= \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - L = 2T - (T-V)$$

$$H = T + V = E \text{ (total energy)} \tag{3.71}$$

When condition (ii) is not satisfied, the Hamiltonian *H* is no longer equal to the total energy of the system. However, the total energy is still conserved for a conservative system.

Summing up, the laws of conservation of linear momentum, angular momentum and total energy are an immediate consequence of the symmetry properties of space and time. For a closed system, there are always seven constants or integrals of motion: linear momentum (3 components), angular momentum
(3 components) and total energy. An interesting point to be noted is that the following pairs of variables are associated with each other: (**r**, **p**) (*q*, **L**) (*t*, *E*) These are the important pairs that follow the uncertainty principle in quantum mechanics.

3.10 VELOCITY-DEPENDENT POTENTIAL

Lagrange's equation in the form as in Eq. (3.38) with **L** = *T* – *V* is applicable only for conservative systems. For systems having non-conservative forces, Lagrange's equations can be put in the same form if the generalized forces are obtained from a function $U(q_j, \dot{q}_j)$ such that $Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$

(3.72) and the Lagrangian is defined by *L* = *T* – *U* (3.73) The potential U is known as a **generalized potential** or **velocity-dependent potential**. An important example of such a potential is the one that gives the electromagnetic forces on moving charges. The force **F** experienced by a charge *q* moving with velocity **v** in an electromagnetic field is given by the Lorentz force: **F =** *q***[E + v B]** (3.74) where the electric intensity **E** and magnetic induction **B** are obtainable from the vector potential **A** and scalar potential *f*:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{3.75}$$

In terms of $\mathbf{A}$ and ϕ, the Lorentz force becomes

$$\mathbf{F} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \times (\nabla \times \mathbf{A}))\right] \tag{3.76}$$

Let us now consider only the x-component of the force $\mathbf{F}$

$$\nabla \times \mathbf{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$(\mathbf{v} \times (\nabla \times \mathbf{A}))_x = \mathrm{v}_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \mathrm{v}_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

Adding and subtracting the term $\mathrm{v}_x \dfrac{\partial A_x}{\partial x}$

$$(\mathbf{v} \times (\nabla \times \mathbf{A}))_x = \mathrm{v}_x \frac{\partial A_x}{\partial x} + \mathrm{v}_y \frac{\partial A_y}{\partial x} + \mathrm{v}_z \frac{\partial A_z}{\partial x} - \mathrm{v}_x \frac{\partial A_x}{\partial x} - \mathrm{v}_y \frac{\partial A_x}{\partial y} - \mathrm{v}_z \frac{\partial A_x}{\partial z}$$

As the x-component of the vector potential $A_X = A_X\,(x, y, z, t)$, the total time derivative of A_X is

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x} v_x + \frac{\partial A_x}{\partial y} v_y + \frac{\partial A_x}{\partial z} v_z + \frac{\partial A_x}{\partial t}$$

Now the x-component of $\mathbf{v} \times (\nabla \times \mathbf{A})$ can be written as

$$\left(\mathbf{v} \times (\nabla \times \mathbf{A})\right)_x = \frac{\partial}{\partial x}(\mathbf{v}.\mathbf{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \tag{3.77}$$

With this substitution the x-component of Lorentz force is

$$F_x = q\left[\frac{-\partial \phi}{\partial x} + \frac{\partial}{\partial x}(\mathbf{v}.\mathbf{A}) - \frac{dA_x}{dt}\right]$$

Since
$$\frac{dA_x}{dt} = \frac{d}{dt}\left(\frac{\partial(\mathbf{v}.\mathbf{A})}{\partial v_x}\right) = \frac{d}{dt}\left(\frac{\partial(\mathbf{v}.\mathbf{A})}{\partial \dot{x}}\right)$$

$$F_x = q\left[\frac{-\partial}{\partial x}\left(\phi - (\mathbf{v}.\mathbf{A})\right) - \frac{d}{dt}\left(\frac{\partial(\mathbf{v}.\mathbf{A})}{\partial v_x}\right)\right] \tag{3.78}$$

Since the scalar potential *f* is independent of velocity, this expression for F_X can be written as $F_x = \frac{d}{dt}\frac{\partial U}{\partial \dot{x}} - \frac{\partial U}{\partial x}$ (3.79)

where,

$U = q\phi - q(\mathbf{v}.\mathbf{A})$ (3.80) U is a generalized potential and the Lagrangian L for a charged particle in an electromagnetic field can be written as $L = T - U = T - q\phi + q\ \mathbf{v}.\mathbf{A}$ (3.81) Since $T = (½)m\dot{x}^2$, the generalized momentum of the charged particle is given by $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x$

or **p** = m**v** + q**A** (3.82) which shows that a part of the momentum is associated with the electromagnetic field.

3.11 DISSIPATIVE FORCE

Often the forces acting on the particle are such that its one part is conservative and the other part is dissipative, like a frictional force which is often proportional to the velocity of the particle. Its x-component $F_x = - k_x v_x$ (3.83) The form of the equations of motion of such a system is mentioned in Eq. (3.39). Frictional forces of the above type may be expressed in terms of a function $G(\mathbf{v})$, called **Rayleigh's dissipation function**: $G(\mathbf{v}) = \frac{1}{2}\sum_{i=1}^{N}\left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right)$ (3.84) where the i summation is over the particles of the system. It is evident from Eq. (3.84) that $F_x = -\frac{\partial G}{\partial v_x}$

In three dimensions

$\mathbf{F} = -\nabla_v G(\mathbf{v})$ (3.85) where ∇_v is the gradient operator in velocity space.

The component of the generalized force resulting from the force of friction is

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_v G \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Using Eq. (3.24)

$Q_j = -\sum_i \nabla_v G \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = -\frac{\partial G}{\partial \dot{q}_j}$ (3.86) Now, Lagrange's equations of motion in the presence of such frictional force is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial G}{\partial \dot{q}_j} = 0$ (3.87) It may be noted that in addition to the Lagrangian L, another function G must also be specified to get the equations of motion.

3.12 NEWTONIAN AND LAGRANGIAN FORMALISMS

The Lagrangian formalism is not the result of a new theory but it is derivable from Newton's second law. In the Newtonian formalism, all the forces acting on the system, both active and internal forces, must be taken into account. That is, the dynamical conditions must be known. But the Lagrangian method concentrates solely on active forces, completely ignoring the forces of constraints by formulating the problem in terms of generalized coordinates. This gives the advantage of selecting any suitable quantity such as linear momentum, angular momentum, velocity or angle as the generalized coordinates depending on the problem. Secondly, the Newtonian force-momentum approach is vectorial in nature whereas the work-energy approach of Lagrangian method involves only scalar functions. In the Lagrangian approach, all the details are contained in a single scalar function, the Lagrangian of the system. Though the directional properties of the vectors are more helpful when dealing with simple systems, the formulation becomes difficult when the system becomes more complex.

The Lagrangian method is applicable to conservative forces only, though procedures are available to study velocity-dependent problems. However, Newtonian mechanics is applicable for both conservative and non-conservative forces.

WORKED EXAMPLES

Example 3.1 Consider a system of N particles with masses m_1, m_2, m_3,...,m_N, located at cartesian coordinates $\mathbf{r}_1$, $\mathbf{r}_2$,..., $\mathbf{r}_N$, acted upon by forces derivable from a potential function V ($\mathbf{r}_1$, $\mathbf{r}_2$,..., $\mathbf{r}_N$). Show that Lagrange's equations of motion reduce directly to Newton's second law.

Solution: The kinetic energy $T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$

Lagrangian $\quad L = T - V = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^{\,2} - V(\mathbf{r}_1, \mathbf{r}_2, \ldots\ldots, \mathbf{r}_N)$

$$\frac{\partial L}{\partial \mathbf{r}_i} = -\frac{\partial V}{\partial \mathbf{r}_i} \qquad \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m\dot{\mathbf{r}}_i \qquad F_i = -\frac{\partial V}{\partial \mathbf{r}_i}$$

Identifying the rectangular co-ordinates as the generalized co-ordinates, Lagrange's equation, Eq. (3.38), can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) - \frac{\partial L}{\partial \mathbf{r}_i} = 0 \quad i = 1, 2, \ldots\ldots, N$$

Substituting the above values

$$\frac{d}{dt}(m_i\,\dot{\mathbf{r}}_i) + \frac{\partial V}{\partial \mathbf{r}_i} = 0 \quad i = 1, 2, \ \ldots\ldots, N$$

$$m_i\,\ddot{\mathbf{r}}_i = -\frac{\partial V}{\partial \mathbf{r}_i} = \mathbf{F}_i \quad i = 1, 2, \ \ldots., N$$

which is the familiar form of Newton's second law.

Example 3.2 Consider a particle moving in space. Using the spherical polar coordinates (*r*, *q*, *f*) as the generalized coordinates, express the virtual displacements *dx*, *dy* and *dz* in terms of *r*, *q* and *f*.

Solution: In terms of coordinates (*r*, *q*, *f*)

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial x}{\partial \phi} = -\,r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial \theta} = -\,r\sin\theta$$

In terms of the generalized coordinates (*r*, *q*, *f*), we have

$$x = x(r,\ \theta,\ \phi) \qquad y = y(r,\ \theta,\ \phi) \qquad z = z(r,\ \theta,\ \phi)$$

$$\delta x = \frac{\partial x}{\partial r}\,\delta r + \frac{\partial x}{\partial \theta}\,\delta\theta + \frac{\partial x}{\partial \phi}\,\delta\phi$$

$$= \sin\theta\cos\phi\,\delta r + r\cos\theta\cos\phi\,\delta\theta - r\sin\theta\sin\phi\,\delta\phi$$

$$\delta y = \frac{\partial y}{\partial r}\,\delta r + \frac{\partial y}{\partial \theta}\,\delta\theta + \frac{\partial y}{\partial \phi}\,\delta\phi$$

$$= \sin\theta\sin\phi\,\delta r + r\cos\theta\sin\phi\,\delta\theta + r\sin\theta\cos\phi\,\delta\phi$$

$$\delta z = \frac{\partial z}{\partial r}\,\delta r + \frac{\partial z}{\partial \theta}\,\delta\theta + \frac{\partial z}{\partial \phi}\,\delta\phi$$

$$= \cos\theta\,\delta r - r\sin\theta\,\delta\theta$$

Example 3.3 A light inextensible string with a mass M at one end passes over a pulley at a distance a from a vertically fixed rod. At the other end of the string is a ring of mass $m(M > m)$ which slides smoothly on the vertical rod as shown in Fig. 3.2. The ring is released from rest at the same level as the point from which the pulley hangs. If b is the maximum distance the ring will fall, determine b using the principle of virtual work.

Solution: Let l be the length of the string. From Fig. 3.2

$$x + \left(a^2 + b^2\right)^{1/2} = l$$

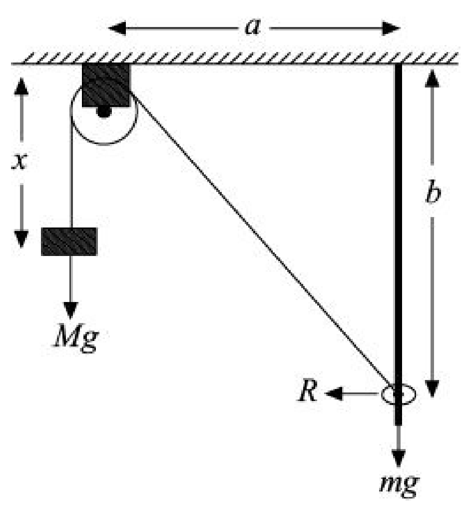


Fig. 3.2 Pulley-ring system with mass M at one end of the string.

Imagine a vertical displacement *db* of the ring along the rod

$$\delta x + b\left(a^2 + b^2\right)^{-1/2} \delta b = 0$$

$$\delta x = -b\left(a^2 + b^2\right)^{-1/2} \delta b$$

The constraints over the pulley and rod do no work. By the principle of virtual work, Eq. (3.15),

$$Mg\ \delta x + mg\ \delta b = 0$$

Substituting the value of δx

$$-Mgb\left(a^2 + b^2\right)^{-1/2} \delta b = -mg\ \delta b$$

Since δb is arbitrary

$$b^2 = \frac{m^2}{M^2}\left(a^2 + b^2\right)$$

$$b = \frac{ma}{\left(M^2 - m^2\right)^{1/2}}$$

Example 3.4 Consider the motion of a particle of mass *m* moving in space. Selecting the cylindrical coordinates (*r*, *f*, *z*) as the generalized coordinates, calculate the generalized force components if a force **F** acts on it.

Solution: The generalized force, Eq. (3.26), corresponding to the coordinate q_j

$$Q_j = \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = F_x \frac{\partial x}{\partial q_j} + F_y \frac{\partial y}{\partial q_j} + F_z \frac{\partial z}{\partial q_j}$$

In cylindrical co-ordinates

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \quad \text{and} \quad z = z$$

$$\frac{\partial x}{\partial \rho} = \cos\phi \qquad \frac{\partial x}{\partial \phi} = -\rho\sin\phi \qquad \frac{\partial x}{\partial z} = 0$$

$$\frac{\partial y}{\partial \rho} = \sin\phi \qquad \frac{\partial y}{\partial \phi} = \rho\cos\phi \qquad \frac{\partial y}{\partial z} = 0$$

$$\frac{\partial z}{\partial \rho} = 0 \qquad \frac{\partial z}{\partial \phi} = 0 \qquad \frac{\partial z}{\partial z} = 1$$

Substituting these values in the expression for generalized force, we have

$$Q_\rho = F_x \frac{\partial x}{\partial \rho} + F_y \frac{\partial y}{\partial \rho} + F_z \frac{\partial z}{\partial \rho}$$

$$= F_x \cos\phi + F_y \sin\phi = F_\rho$$

$$Q_\phi = -F_x \rho \sin\phi + F_y \rho \cos\phi = \rho F_\phi$$

$$Q_z = F_z$$

where F_r, F_f and F_Z are the components of the force along the increasing directions of *r*, *f* and *z*.

Example 3.5 Find Lagrange's equation of motion of the bob of a simple pendulum.

Solution: Let us select the angle *q* made by the string with the vertical axis as the generalized coordinate as shown in Fig. 3.3. Since *l* is a constant, kinetic energy of the bob $T = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$

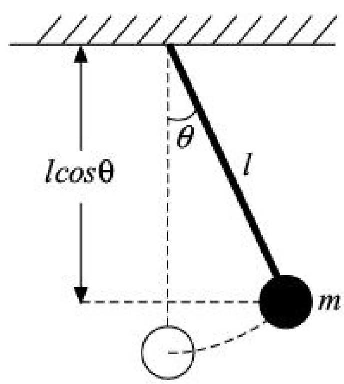


Fig. 3.3 Simple pendulum.

Taking the mean position of the bob as the reference point

The potential energy of the bob $V = mg\left(l - l\cos\theta\right)$

The Lagrangian $$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl\left(1 - \cos\theta\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = -\, mgl \sin\theta$$

Substituting these quantities in Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml^2 \ddot{\theta} + mgl \sin\theta = 0$$

$$\ddot{\theta} + \left(g/l\right) \sin\theta = 0$$

which is Lagrange's equation of motion of the bob of a simple pendulum.

Example 3.6 Obtain the equations of motion for the motion of a particle of mass *m* in a potential *V* (*x*, *y*, *z*) in spherical polar coordinates.

Solution: In spherical polar coordinates the elementary lengths are *dr*, *rdq*, *r* sin*q df* and velocities are $\dot{r}, r\dot{\theta}, r\sin\theta\dot{\phi}$

Kinetic energy $$T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right)$$

Lagrangian $$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2\right) - V\left(r, \theta, \phi\right)$$

Lagrange's equations of motion: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1, 2, 3, \ldots$$

Identifying *r*, *q* and *f* as as the generalized coordinates, the equations of motion are *r* coordinate:

$$\frac{d(m\dot{r})}{dt} - mr\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + \frac{\partial V}{\partial r} = 0$$

θ co-ordinate:

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 + \frac{\partial V}{\partial\theta} = 0$$

ϕ co-ordinate:

$$\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\ \right) + \frac{\partial V}{\partial\phi} = 0$$

Note: If the force is central, $\quad \dfrac{\partial V}{\partial\theta} = \dfrac{\partial V}{\partial\phi} = 0$

Example 3.7 Masses *m* and 2*m* are connected by a light inextensible string which passes over a pulley of mass 2*m* and radius *a*. Write the Lagrangian and find the acceleration of the system.

Solution: The system has only one degree of freedom, and *x* (see Fig. 3.4) is taken as the generalized coordinate. The length of the string be *l* and the centre of the pulley is taken as zero for potential energy.

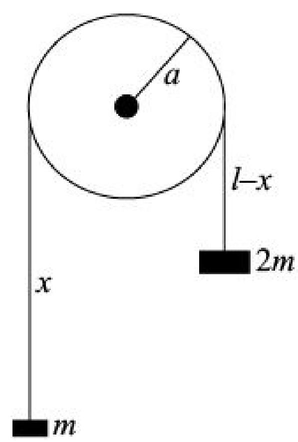


Fig. 3.4 A pulley with a string carrying masses *m* and 2*m* at its end.

K.E. of the system $$T = \frac{1}{2} m\dot{x}^2 + m\dot{x}^2 + \frac{1}{2} I\omega^2$$

$$= \frac{3}{2} m\dot{x}^2 + \frac{1}{2} I\left(\frac{\dot{x}}{a}\right)^2$$

P.E. of the system $V = -mgx - 2mg\,(l - x)$

Lagrangian $$L = \frac{3}{2} m\dot{x}^2 + \frac{I}{2a^2}\dot{x}^2 - mgx + 2mgl$$

$$\frac{\partial L}{\partial \dot{x}} = \left(3m + \frac{I}{a^2}\right)\dot{x} \qquad \frac{\partial L}{\partial x} = -mg$$

Substitution of these derivatives in Lagrange's equation gives the equation of motion:

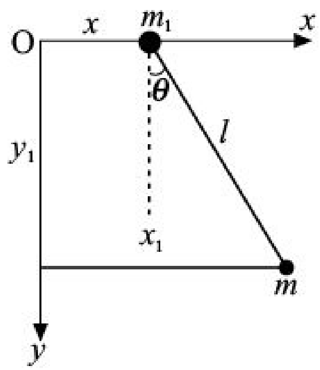
$$\left(3m + \frac{I}{a^2}\right)\ddot{x} + mg = 0$$

Acceleration $$\ddot{x} = -\frac{mg}{(3m + I/a^2)} = -\frac{g}{4}$$

since moment of inertia of the disc $= \frac{1}{2} \times 2ma^2 = ma^2$. Minus sign indicates mass m moves upwards with the acceleration $g/4$.

Example 3.8 A simple pendulum has a bob of mass m with a mass m_1 at the moving support (pendulum with moving support) which moves on a horizontal line in the vertical plane in which the pendulum oscillates. Find the Lagrangian and Lagrange's equation of motion.

Solution: This pendulum (see Fig. 3.5) has two degrees of freedom, and x and q can be taken as the generalized coordinates. Taking the point of support as the



zero of potential energy

Fig. 3.5 Simple pendulum with a moving support.

$$\text{P.E. of mass} \quad m_1 = 0$$

$$\text{K.E. of mass} \quad m_1 = \frac{1}{2} m_1 \dot{x}^2$$

$$\text{K.E. of mass} \quad m = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{y}_1^2 \right)$$

$$= \frac{1}{2} m \left[\frac{d}{dt} (x + l \sin\theta) \right]^2 + \frac{m}{2} \left[\frac{d(l \cos\theta)}{dt} \right]^2$$

$$= \frac{m}{2}\left[\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right]$$

P.E. of $m = -mgl\cos\theta$

$$L = T - V$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + mgl\cos\theta$$

Lagrange's equation of motion for x is

$$\frac{d}{dt}\left\{(m + m_1)\,\dot{x} + ml\cos\theta\,\dot{\theta}\right\} = 0$$

$$(m + m_1)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = 0$$

Equation of motion for θ is

$$\frac{d}{dt}\left(ml^2\dot{\theta} + ml\cos\theta\,\dot{x}\right) + ml\sin\theta\,\dot{x}\dot{\theta} + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + ml\cos\theta\ddot{x} + mgl\sin\theta = 0$$

Example 3.9 Two equal masses *m* connected by a massless rigid rod of length *l* forming a dumb-bell is rotated in the *x*-*y* plane. Find the Lagrangian and obtain Lagrange's equations of motion.

Solution: Figure 3.6 illustrates the motion of the dumb-bell in the *x*-*y* plane.

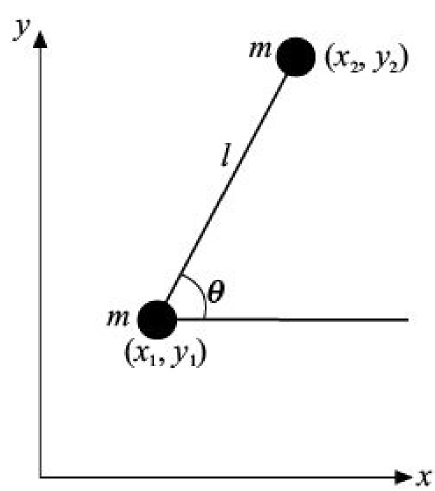


Fig. 3.6 Dumb-bell in the *x*-*y* plane.

The system has 3 degrees of freedom. The cartesian coordinates x_1, y_1 and the angle q can be selected as the generalized coordinates. From the figure $x_2 = x_1 + l \cos q$ $y_2 = y_1 + l \sin q$

Kinetic energy, $T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right)$

$$= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m\left[\left(\dot{x}_1 - l\sin\theta\,\dot{\theta}\right)^2 + \left(\dot{y}_1 + \ l\cos\theta\,\dot{\theta}\right)^2\right]$$

$$= m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m\left(l^2\dot{\theta}^2 - 2\,l\dot{x}_1\dot{\theta}\sin\theta + 2l\dot{y}_1\dot{\theta}\cos\theta\right)$$

Taking x-axis as the reference level, the potential energy is given by

$$V = mgy_1 + mgy_2 = 2mgy_1 + mgl\sin\theta$$

$$L = m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m\left(l^2\dot{\theta}^2 - 2l\dot{x}_1\dot{\theta}\,\sin\theta + 2\,l\dot{y}_1\,\dot{\theta}\cos\theta\right) - 2mgy_1 - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}_1} = 2m\,\dot{x}_1 - ml\dot{\theta}\sin\theta \qquad \frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial \dot{y}_1} = 2\,m\,\dot{y}_1 + ml\,\dot{\theta}\cos\theta \qquad \frac{\partial L}{\partial y_1} = -\,2mg$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - ml\,\dot{x}_1\,\sin\theta + ml\,\dot{y}_1\,\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -\,ml\,\dot{x}_1\dot{\theta}\cos\theta - ml\,\dot{y}_1\dot{\theta}\sin\theta - mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 2m\,\ddot{x}_1 - m\,l\ddot{\theta}\,\sin\theta - ml\dot{\theta}^2\,\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = 2m\,\ddot{y}_1 - m\,l\ddot{\theta}\cos\theta - ml\dot{\theta}^2\,\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\,\ddot{\theta} - ml\,\ddot{x}_1\sin\theta - ml\,\dot{x}_1\,\dot{\theta}\cos\theta + ml\,\ddot{y}_1\cos\theta - ml\,\dot{y}_1\,\dot{\theta}\sin\theta$$

Substitution of the above quantities in the Lagrange's equation gives the following equations of motion:

$$2\ddot{x}_1 - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\,\cos\theta = 0$$

$$2\ddot{y}_1 + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\,\sin\theta = 0$$

$$l^2\ddot{\theta} - l\,\ddot{x}_1\sin\theta + l\,\ddot{y}_1\cos\theta + gl\cos\theta = 0$$

Example 3.10 A simple pendulum that is free to swing the entire solid angle is called a *spherical pendulum*. Find the differential equations of motion of a spherical pendulum using Lagrange's method. Also show that the angular momentum about a vertical axis through the point of support is a constant of motion.

Solution: Figure 3.7 illustrates the motion of the spherical pendulum in spherical polar coordinates. Since l is constant, the system has two degrees of freedom. Angles q and f can be selected as the generalized coordinates.

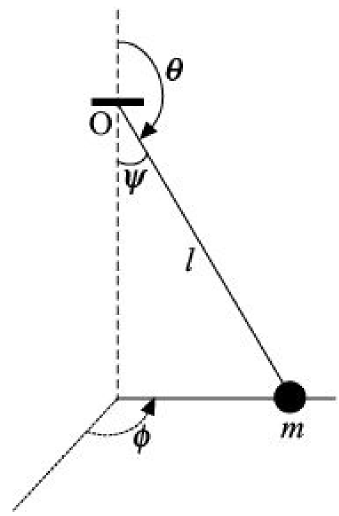


Fig. 3.7 Spherical pendulum.

Taking the point of support as the reference level for potential energy V, we

$$V = -\,mgl\cos\psi = mgl\cos\theta$$

Elementary lengths are $ld\theta$ and $l\sin\psi\, d\phi = l\sin\theta\, d\phi$

Kinetic energy

$$T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2\right)$$

Lagrangian

$$L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta$$

have

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \qquad \frac{\partial L}{\partial\theta} = ml^2\dot{\phi}^2\sin\theta\cos\theta + mgl\sin\theta$$

$$\frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi}\sin^2\theta \qquad \frac{\partial L}{\partial\phi} = 0$$

Lagrange's equation for the co-ordinate θ is

$$ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta - mgl\sin\theta = 0$$

For the co-ordinate ϕ

$$ml^2\ddot{\phi}\sin^2\theta + 2ml^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0$$

Since $(\partial L/\partial\phi) = 0$, from Lagrange's equation for ϕ, we have

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = 0 \qquad \text{or} \qquad \frac{d}{dt}(ml^2\sin^2\theta\dot{\phi}) = 0$$

$$ml^2\sin^2\theta\dot{\phi} = \text{constant}$$

In spherical polar coordinates, the magnitude of the angular momentum is $m(l\sin\theta\dot{\phi})\,l\sin\theta = ml^2\sin^2\theta\dot{\phi}$, which is the same as the above. Hence, the angular momentum about a vertical axis through the point of support is a constant of motion.

Example 3.11 A bead of mass *m* slides freely on a frictionless circular wire of radius *a* that rotates in a horizontal plane about a point on the circular wire with a constant angular velocity *w*. Find the equation of motion of the bead by

Lagrange's method. Also show that the bead oscillates as a pendulum of length *l* = *g*/*w*^{2}

Solution: The circular wire rotates in the *x*-*y* plane about the point O in the counterclockwise direction with an angular velocity *w*. *A* is the centre of the circular wire. The angles *f* and *q* are as indicated in Fig. 3.8. The coordinates of *m* are (*x*, *y*). The problem is of one degree of freedom and *q* can be taken as the generalized coordinate. The potential energy of the bead can be taken as zero since the circular wire is in a horizontal plane.

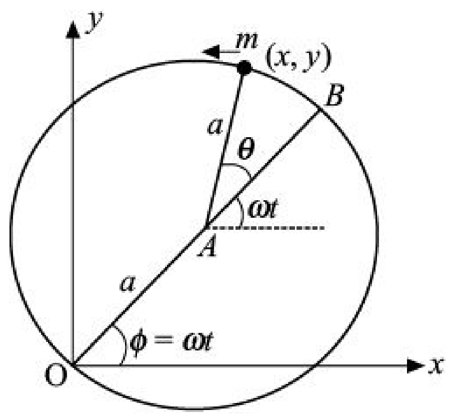


Fig. 3.8 A bead sliding on a circular wire.

$$x = a\cos\omega t + a\cos(\theta + \omega t)$$

$$y = a\sin\omega t + a\sin(\theta + \omega t)$$

$$\dot{x} = -a\omega\sin\omega t - a(\dot{\theta} + \omega)\sin(\theta + \omega t)$$

$$\dot{y} = a\omega\cos\omega t + a(\dot{\theta} + \omega)\cos(\theta + \omega t)$$

Lagrangian

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m a^2 \left[\omega^2 + \left(\dot{\theta} + \omega\right)^2 + 2\omega\left(\dot{\theta} + \omega\right)\cos\theta \right]$$

$$\frac{\partial L}{\partial \dot{\theta}} = m a^2 \left(\dot{\theta} + \omega + \omega \cos\theta\right) \qquad \frac{\partial L}{\partial \theta} = -\, m\omega a^2 \left(\dot{\theta} + \omega\right)\sin\theta$$

Substituting these values in Lagrange's equation, we get

$$m a^2 \left(\ddot{\theta} - \omega\dot{\theta}\sin\theta\right) + m a^2 \omega\left(\dot{\theta} + \omega\right)\sin\theta = 0$$

or

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

which is the equation of motion of the mass *m*. Comparing the equation of motion with that of the simple pendulum (see Example 3.5) we see that the bead oscillates about the line OAB like a pendulum of length $l = g/\omega^2$.

Example 3.12 A particle of mass *m* is constrained to move on the inner surface of a cone of half angle *a* with its apex on a table. Obtain its equation of motion in cylindrical coordinates (*r*, *f*, *z*). Hence, show that the angle *f* is a cyclic coordinate.

Solution: As the particle is moving on the surface of the cone, the equation of constraint is $\rho = z\tan\alpha$ or $z = \rho\cot\alpha$

Since there is an equation of constraint the particle requires only 2 generalized coordinates, say ρ and ϕ (see Fig. 3.9).

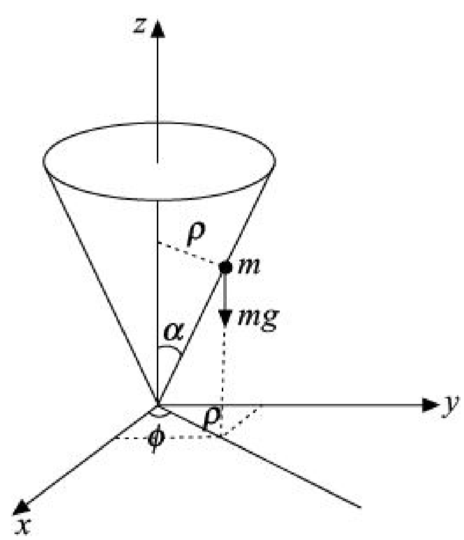


Fig. 3.9 Mass *m* moving on the inner surface of a cone.

Kinetic energy $T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right)$

$$= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{\rho}^2 \cot^2 \alpha\right)$$

$$= \frac{1}{2}m\left(\rho^2\dot{\phi}^2 + \dot{\rho}^2 \operatorname{cosec}^2 \alpha\right)$$

Assuming that the potential energy $V = 0$ at $z = 0$

$$V = mgz = mg\rho \cot\alpha$$

Lagrangian

$$L = \frac{1}{2}m\left(\rho^2\dot{\phi}^2 + \dot{\rho}^2 \operatorname{cosec}^2 \alpha\right) - mg\rho \cot\alpha$$

$$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \operatorname{cosec}^2 \alpha \qquad \frac{\partial L}{\partial \rho} = m\rho\dot{\phi}^2 - mg \cot\alpha$$

$$\frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \qquad \frac{\partial L}{\partial \phi} = 0$$

Lagrange's equation for the co-ordinate ρ is

$$\ddot{\rho} \operatorname{cosec}^2\alpha - \rho\dot{\phi}^2 + g\cot\alpha = 0$$

Lagrange's equation for the co-ordinate ϕ is

$$\frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = 0 \quad \text{or} \quad m\rho^2\dot{\phi} = \text{constant}$$

The generalized momentum corresponding to the generalized co-ordinate ϕ is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = \text{constant}$$

Since the generalized momentum is a constant, the corresponding coordinate is a cyclic one. This can be seen from the Lagrangian itself which is independent of *f*.

Example 3.13 An inclined plane of mass *M* is sliding on a smooth horizontal surface, while a particle of mass *m* is sliding on the smooth inclined plane. Find the equation of motion of the particle and that of the inclined plane.

Solution: The system has two degrees of freedom. Let x_1 be the displacement of

M from origin O and x_2 be the displacement of m fromO [see Fig. 3.10(a)].
We shall consider x_1 and x_2 as the generalized coordinates.

The velocity of *M* with respect to O is $\dot{x}_1$.

The velocity of *m* with respect to O is $\dot{x}_2$.

The velocity of *m* with respect to O [see Fig. 3.10(b)] is given by

$$\mathbf{v} = \dot{\boldsymbol{x}}_1 + \dot{\boldsymbol{x}}_2$$

$$\mathbf{v}^2 = \dot{x}_1^2 + \dot{x}_2^2 + 2\,\dot{x}_1\,\dot{x}_2\cos\theta \qquad \text{(i)}$$

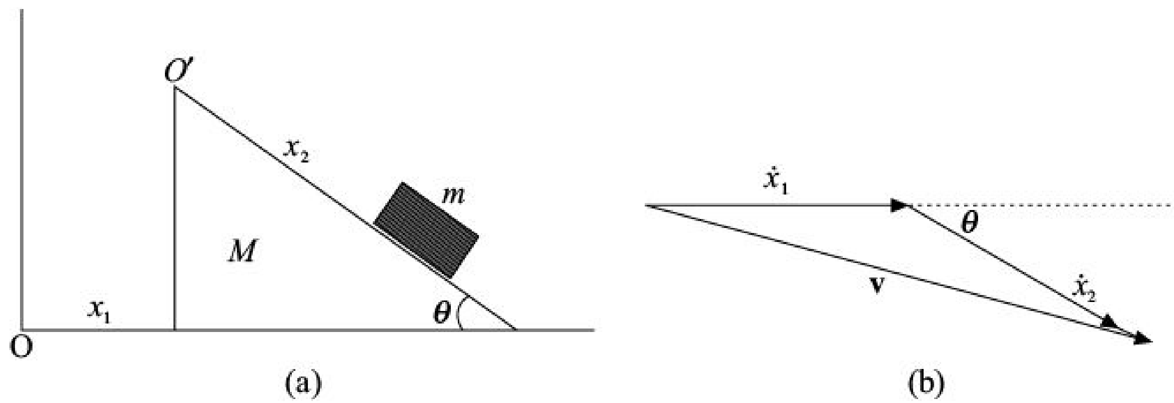


Fig 3.10 A particle sliding on an inclined plane which is sliding on a horizontal surface.

Kinetic energy $\quad T = \frac{1}{2}M\,\dot{x}_1^2 + \frac{1}{2}m\mathbf{v}^2$

$$= \frac{1}{2}(M+m)\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + m\dot{x}_1\,\dot{x}_2\cos\theta \qquad \text{(ii)}$$

If *O* is taken as the zero for potential energy, the potential energy of *M* will be a constant which will not affect the motion. The potential energy of *m* is given by $\quad V = -\,mg\,x_2\sin\theta \qquad$ (iii)

From Eqs. (ii) and (iii)

$$L = T - V = \frac{1}{2}(M+m)\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + m\dot{x}_1\dot{x}_2 \cos\theta + mgx_2 \sin\theta \qquad \text{(iv)}$$

$$\frac{\partial L}{\partial \dot{x}_1} = (M+m)\dot{x}_1 + m\dot{x}_2 \cos\theta \qquad \frac{\partial L}{\partial x_1} = 0 \qquad \text{(v)}$$

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 + m\dot{x}_1 \cos\theta \qquad \frac{\partial L}{\partial x_2} = mg\sin\theta \qquad \text{(vi)}$$

Lagrange's equations for x_1 and x_2 are

$$(M+m)\ddot{x}_1 + m\ddot{x}_2 \cos\theta = 0 \qquad \text{(vii)}$$

$$m\ddot{x}_1 \cos\theta + m\ddot{x}_2 - mg\sin\theta = 0 \qquad \text{(viii)}$$

Solving Eqs. (vii) and (viii)

$$\ddot{x}_1 = -\frac{g\sin\theta\cos\theta}{\left(1+\frac{M}{m}\right) - \cos^2\theta} \quad \text{and} \quad \ddot{x}_2 = \frac{g\sin\theta}{1-\left(\frac{m}{M+m}\right)\cos^2\theta} \qquad \text{(ix)}$$

Equation (ix) is the equation of motion of the inclined plane and of the body sliding down the inclined plane respectively.

Example 3.14 A rigid body capable of oscillating in a vertical plane about a fixed horizontal axis is called a *compound pendulum*. (i) Set up its Lagrangian; (ii) Obtain its equations of motion; and (iii) Find the period of the pendulum.

Solution: Let the vertical plane of oscillation be *xy*. Let the point *O* be the axis of oscillation, *m* be the mass of the body, *G* its centre of mass and *I* its moment of inertia about the axis of oscillation. The system has only one degree of freedom.

(i) Angle *q* can be taken as the generalized coordinate (see Fig.3.11).

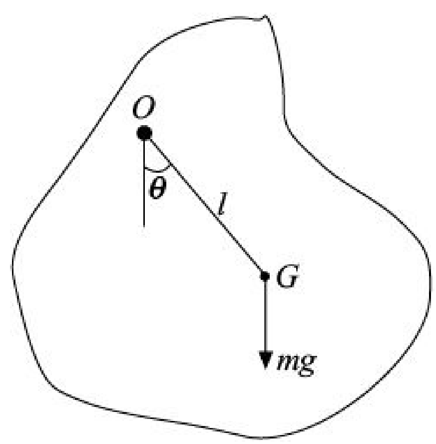


Fig. 3.11 Compound pendulum.

When the displacement is *q*, the kinetic energy $T = \frac{1}{2} I \dot{\theta}^2$

With respect to the point of oscillation, the potential energy

$$V = -\, mgl \cos\theta$$

Lagrangian $\quad L = T - V = \frac{1}{2} I \dot{\theta}^2 + mgl \cos\theta$

(ii) Lagrange's equation for θ is

$$\frac{d\left(I\dot{\theta}\right)}{dt} + mgl \sin\theta = 0$$

$$I\ddot{\theta} + mgl \sin\theta = 0$$

$$\ddot{\theta} = -\frac{mgl}{I} \sin\theta$$

(iii) For small oscillations, $\sin\theta \cong \theta$. Hence,

$$\ddot{\theta} = -\frac{mgl\,\theta}{I}$$

which is the equation of a simple harmonic motion. Hence, period

$$T = 2\pi\sqrt{\frac{I}{mgl}} \qquad I = mk^2 + ml^2 \qquad k = \text{radius of gyration}$$

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

Example 3.15 A mass *M* is suspended from a spring of mass *m* and spring constant *k*. Write the Lagrangian of the system and show that it executes simple harmonic motion in the vertical direction. Also, obtain an expression for its period of oscillation.

Solution: The direction of motion of the mass is selected as the *x*-axis as illustrated in Fig. 3.12. The velocity of the spring at the end where the mass *M* is attached is maximum, say $\dot{x}$ and minimum (zero) at $x = 0$. At the distance *t* from the fixed end, the velocity is $(t/l)\dot{x}$, where *l* is the length of the spring. If *r* is the mass per unit length of the spring, the kinetic energy of the element of length *dt* is $$dT = \frac{1}{2}\rho\, dt\left(\frac{t}{l}\dot{x}\right)^2$$

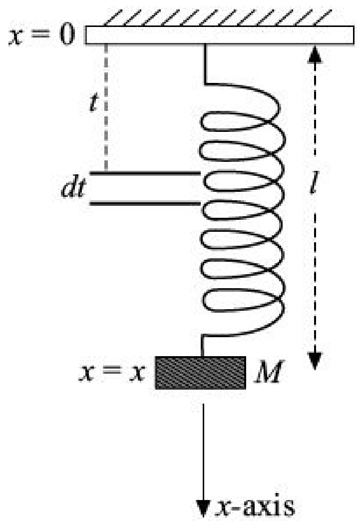


Fig. 3.12 Vibration of a loaded spring of mass *m*.

For the whole spring

$$T = \frac{1}{2}\int_0^l \rho\, dt \frac{t^2}{l^2}\dot{x}^2 = \frac{1}{2}\frac{\rho\,\dot{x}^2}{l^2}\int_0^l t^2 dt = \frac{\rho\,\dot{x}^2\, l}{6}$$

$$= \frac{m\,\dot{x}^2}{6}$$

Total K.E. $$T = \frac{m\,\dot{x}^2}{6} + \frac{1}{2}M\,\dot{x}^2$$

Potential energy $$V = \frac{1}{2}kx^2$$

$$L = \frac{m\,\dot{x}^2}{6} + \frac{1}{2}M\,\dot{x}^2 - \frac{1}{2}kx^2$$

$$= \frac{1}{2}\left(\frac{m}{3} + M\right)\dot{x}^2 - \frac{1}{2}kx^2$$

Lagrange’s equation is

$$\left(\frac{m}{3} + M\right)\ddot{x} + kx = 0$$

which is the equation of simple harmonic motion.

Period $$T = 2\pi\sqrt{\left(\frac{m}{3} + M\right)\Big/k}$$

Example 3.16 A particle of mass *m* is constrained to move on the surface of a cylinder of radius *a*. It is subjected to an attractive force directed towards the origin and is proportional to the distance of the particle from the origin. Write its Lagrangian in cylindrical coordinates and (i) obtain its equations of motion,
(ii) show that the angular momentum about the *z*-axis is a constant of motion, and (iii) show that the motion of the particle in the *z*-direction is simple harmonic.

Solution: The motion of the particle is illustrated in Fig. 3.13. It can be described by the cylindrical coordinates (r, q, z). In the present case $r = a =$ constant. The equation of constraint is $x^2 + y^2 = a^2$
Coordinates q and z can be taken as the generalized coordinates.

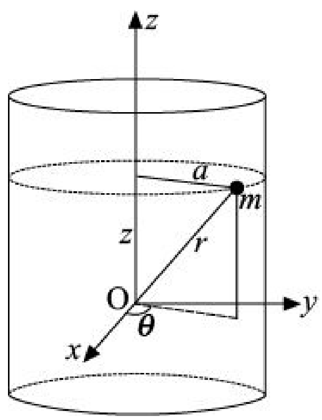


Fig. 3.13 A mass *m* constrained to move on the surface of a cylinder.

$$V = \frac{1}{2} kr^2 = \frac{1}{2} k\ (x^2 + y^2 + z^2)$$

$$= \frac{1}{2} k(a^2 + z^2)$$

Force = –*kr*. Hence,

Kinetic energy $T = \frac{1}{2} m(\dot{a}^2 + a^2\dot{\theta}^2 + \dot{z}^2)$

Since a is constant

$$T = \frac{1}{2} m(a^2\dot{\theta}^2 + \dot{z}^2)$$

$$L = \frac{1}{2} m(a^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k(a^2 + z^2) \qquad \text{(i)}$$

(i) $\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$

Lagrange's equation of motion for co-ordinate θ is

$$\frac{d}{dt}(ma^2\dot{\theta}) = 0 \qquad \text{(ii)}$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \frac{\partial L}{\partial z} = -kz$$

Lagrange's equation for co-ordinate z is

$$\frac{d}{dt}(m\dot{z}) + kz = 0 \qquad \text{or} \qquad \ddot{z} = -\frac{k}{m} z \qquad \text{(iii)}$$

(i) Eqs. (ii) and (iii) are the equations of motion.

(ii) From Eq. (ii) we have

$$ma^2\dot{\theta} = \text{constant}$$

That is, the angular momentum about the z-axis is a constant of motion.

(iii) From Eq. (iii), it is obvious that the motion is simple harmonic with frequency $\omega = \sqrt{k/m}$.

Example 3.17 If L is the Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, q_2, ..., q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary but differentiable function of its arguments.

Solution: For L to satisfy Lagrange's equation of motion, we must have

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n$$

Writing $\dot{F}$ for (dF/dt) and substituting for L'

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - \frac{\partial \dot{F}}{\partial q_j} = 0 \qquad \text{(i)}$$

Since $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$, for L' to satisfy Lagrange's equation, F must satisfy

$$\frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}_j}\right) - \frac{\partial \dot{F}}{\partial q_j} = 0 \qquad \text{(ii)}$$

We have $\qquad F = F(q_1, q_2, \ldots, q_n, t)$

$$\dot{F} = \sum_j \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}$$

$$\frac{\partial \dot{F}}{\partial \dot{q}_j} = \frac{\partial F}{\partial q_j} \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) = \frac{\partial \dot{F}}{\partial q_j} \qquad \text{(iii)}$$

which is the same as Eq. (ii). Hence, the result.

REVIEW QUESTIONS

1. Explain holonomic and non-holonomic constraints, giving two examples of each.
2. Gas molecules are confined to move in a box. What is the type of constraint on the motion of the gas molecules? Explain.
3. What is meant by degrees of freedom? What is the number of degrees of freedom that a body which is constrained to move along a space curve has?

4. Explain the difference between real and virtual displacements. In a virtual displacement, the work done by the forces of constraint is zero. Why?
5. State and explain the principle of virtual work.
6. What are generalized coordinates? If a generalized coordinate has the dimension of momentum, what would be the dimension of generalized velocity?
7. Explain the type of constraint in: (i) a pendulum with an inextensible string; (ii) a pendulum with an extensible string.
8. What is configuration space? Why does the path of motion in the configuration space not necessarily resemble the path in space of an actual particle ?
9. State and explain D' Alembert's principle.
10. What is generalized momentum $\mathbf{p}_j$? For generalized momentum $\mathbf{p}_j$, establish the relation $\dot{\mathbf{p}}_j = \frac{\partial L}{\partial q_j}$.
11. What are the first integrals of motion?
12. What is a cyclic coordinate ? Why do we say that the generalized momentum conjugate to a cyclic coordinate is a constant of motion?
13. The homogeneity of space implies that the linear momentum is a constant of motion. Substantiate.
14. The homogeneity of time implies that the total energy is a constant of motion. Substantiate.
15. What are velocity-dependent potentials?
16. Write the Lagrangian of a charged particle in an electromagnetic field, explaining each term.
17. What is a dissipation function? How is it related to the force it represents?
18. Evaluate the dissipation function corresponding to Stoke's law.

PROBLEMS

1. A particle of mass m is moving in a plane. Using plane polar coordinates as the generalized coordinates, find the displacements dx and dy .
2. Consider the motion of a particle of mass m moving in space. Selecting the cylindrical coordinates (r, f, z) as the generalized coordinates, calculate displacements dr, df and dz.
3. Two blocks of masses m_1 and m_2 are placed on a frictionless double inclined plane and are connected by an inextensible massless string passing over a

smooth pulley at the top of the inclines. Find the condition for equilibrium by the principle of virtual work.

4. A uniform plank of mass *M* and length 2*l* is leaning against a smooth wall and makes an angle *a* with the smooth floor. The lower end of the plank is connected to the base of the wall with an inextensible massless string. Using the principle of virtual work, find the tension in the string.
5. A particle of mass *m* is moving in a plane under the action of a force **F**. Using the generalized coordinates (*r*, *q*), calculate the generalized forces for the particle.
6. Consider the motion of a particle of mass *m* moving in space. Selecting the spherical polar coordinates (*r*, *q*, *f*) as the generalized coordinates, calculate the generalized force components if a force **F** acts on it.
7. A particle of mass *m* is moving in a plane under an inverse square attractive force. Find the equation of motion by the Lagrangian method.
8. A mass is attached to a spring having a spring constant *k* which is suspended from a hook. Set up Lagrange's equation of motion (i) if the mass executes simple harmonic motion, and (ii) if the mass is driven by a sinusoidal force A_0 sin *wt*.
9. A light inextensible string passes over a smooth massless pulley and carries masses m_1 and m_2 ($m_1 > m_2$) at its ends. Write down the Lagrangian and Lagrange's equation of motion for the system. Also find the acceleration.
10. A particle of mass *m* is moving in a potential *V* which is a function of coordinates only. Set up the Lagrangian in cylindrical coordinates and obtain the equations of motion.
11. Set up the Lagrangian of a three-dimensional isotropic harmonic oscillator in polar coordinates and obtain Lagrange's equations of motion.
12. A particle of mass *m* is projected in space with velocity $\mathbf{v}_0$ at an angle a to the horizontal. Write the Lagrangian for the motion of the projectile and show that its path is a parabola. Also find expressions for the range and time of flight.
13. A cylinder of mass *m* and radius *a* rolls down an inclined plane of angle *q*. Write the Lagrangian of the system and obtain the equation of motion. Also, calculate its velocity at the bottom of the plane.
14. In a double pendulum, the second pendulum is suspended from the mass of the first one (m_1) which is suspended from a support. The double pendulum is set into oscillation in a vertical plane. Obtain the Lagrangian and equations of

motion for the double pendulum if the mass of the second one is m_2.

15. A particle of mass *m* moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2\, V(x) - V^2(x)$$

where *V* is a differentiable function of *x*. Find the equation of motion for *x*(*t*) and interpret the physical nature of the system.

16. A solid homogeneous cylinder of radius *a* rolls without slipping inside a stationary hollow cylinder of large radius *R*. Write the Lagrangian and obtain the equation of motion. Also show that the motion of the solid cylinder is simple harmonic and deduce its frequency.

17. A bead of mass *m* slides on a wire described by the equations $x = a(q - \sin q)$, $y = a(1 + \cos q)$ where $0 \le q \le 2p$. Deduce (i) the Lagrangian and (ii) the equation of motion of the system.

18. A mass *M* is suspended from a fixed support by a spring of spring constant k_1. From this mass another mass *m* is suspended by another spring of spring constant *k*. If the respective displacements of masses are x_1 and x_2, obtain the equation of motion of the system.

4

Variational Principle

In Chapter 3, Lagrange's equations of motion were derived from D'Alembert's principle which is a differential principle. In this chapter the basic laws of mechanics are obtained from an integral principle known as *Hamilton's variational principle*. In this procedure, Lagrange's equations of motion are obtained from a statement about the value of the time integral of the Lagrangian between times t_1 and t_2. In D'Alembert's principle, we considered the instantaneous state of the system and virtual displacements from the instantaneous state. However, in the following variational procedure, infinitesimal virtual variations of the entire motion from the actual one is considered.

4.1 HAMILTON'S PRINCIPLE

Hamilton's principle is a variational formulation of the laws of motion in configuration space. It is considered more fundamental than Newton's equations as it can be applied to a variety of physical phenomena.

The configuration of a system at any time is defined by the values of the n generalized co-ordinates $q_1, q_2, q_3, \ldots, q_n$. This corresponds to a particular point in the n-dimensional configuration space in which the q_i's are components along the n co-ordinate axes. Hamilton's principle states: *For a conservative holonomic system, the motion of the system from its position at time t_1 to its position at time t_2 follows a path for which the line integral*

$$I = \int_{t_1}^{t_2} L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)\, dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \qquad (4.1)$$

has a stationary value.

That is, out of all possible paths by which the system point could travel from its position at time t_1 to its position at time t_2 in the configuration space consistent with the constraints, the path followed by the system is that for which the value of the above integral is stationary. Mathematically, the principle can be stated as:

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0 \qquad (4.2)$$

where $q_i(t)$ and hence $\dot{q}_i(t)$ is to be varied such that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The time integral of the Lagrangian L, Eq.(4.1), is called the **action integral** or simply **action**. The *d*-variation considered here refers to the variation in a quantity at the same instant of time (see section 3.3) while the *d*-variation as usual refers to a variation in quantity along a path at different instants of time (see Fig.4.1). The two paths are infinitely close but arbitrary.

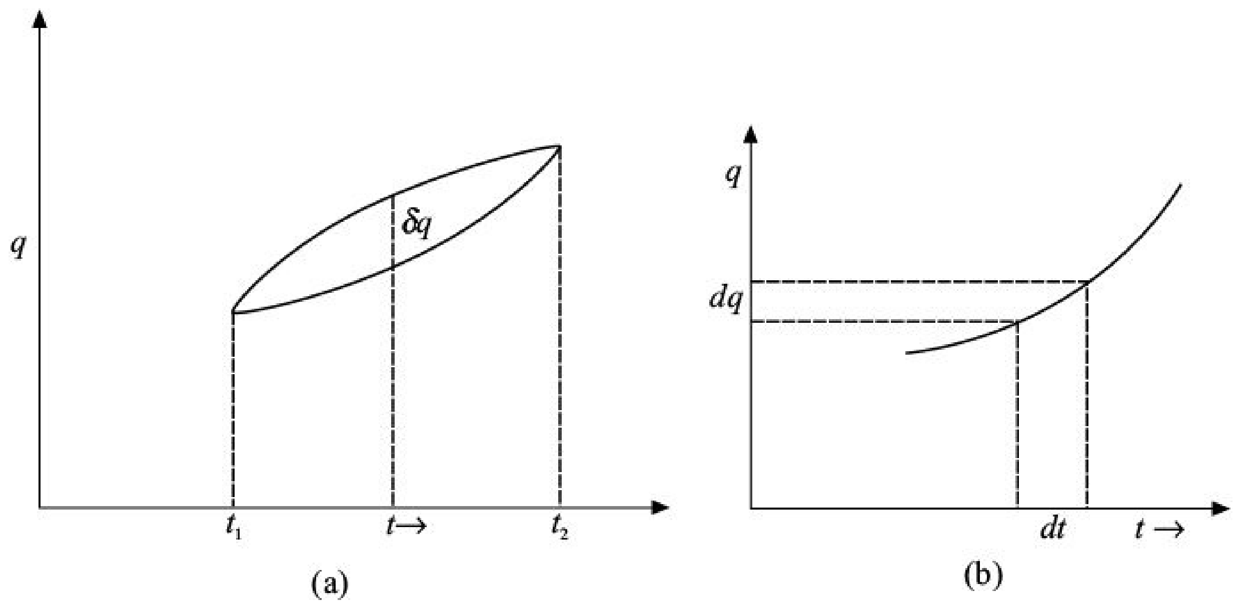


Fig. 4.1 (a) *d*-variation in motion; (b) *d*-variation in motion.

4.2 DEDUCTION OF HAMILTON'S PRINCIPLE

Hamilton's principle can be easily deduced from D'Alembert's principle given by Eq. (3.19). Consider a system of N particles of masses m_i, $i = 1, 2, ..., N$, located at points r_i and acted upon by external forces $\mathbf{F}_i$. According to D'Alembert's principle

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\,\ddot{\mathbf{r}}_i).\delta\mathbf{r}_i = 0 \qquad (4.3)$$

The term $\sum_i \mathbf{F}_i\,.\delta\mathbf{r}_i$ is the virtual work δW done by the external applied forces $\mathbf{F}_i$, its value is given by Eq. (3.25):

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i\,.\delta\mathbf{r}_i = \sum_{j=1}^{n} Q_j \delta q_j \qquad (4.4)$$

where Q_j is the generalized force, defined by Eq. (3.26), and q's are the generalized co-ordinates of the system. The second term in Eq. (4.3) can be written as

$$\sum_i m_i\,\ddot{\mathbf{r}}_i\,.\delta\mathbf{r}_i = \frac{d}{dt}\left(\sum_i m_i\dot{\mathbf{r}}_i\,.\delta\mathbf{r}_i\right) - \sum_i m_i\dot{\mathbf{r}}_i\,\frac{d}{dt}(\delta\mathbf{r}_i)$$

Since virtual displacement is at the same instant of time, the order of δ and d-variations can be interchanged.

$$\sum_i m_i\,\ddot{\mathbf{r}}_i\,.\delta\mathbf{r}_i = \frac{d}{dt}\left(\sum_i m_i\,\dot{\mathbf{r}}_i\,.\delta\mathbf{r}_i\right) - \sum_i m_i\dot{\mathbf{r}}_i\,.\delta\dot{\mathbf{r}}_i$$

$$= \frac{d}{dt}\left(\sum_i m_i\dot{\mathbf{r}}_i\,.\delta\mathbf{r}_i\right) - \delta\left(\frac{1}{2}\sum_i m_i\dot{\mathbf{r}}_i\,.\dot{\mathbf{r}}_i\right) \qquad (4.5)$$

The second term on the right hand side of Eq. (4.5) is the d-variation of kinetic energy T. Now Eq. (4.5) takes the form

$$\sum_i m_i \ddot{\mathbf{r}}_i \,.\delta\mathbf{r}_i = \frac{d}{dt}\sum_i m_i \dot{\mathbf{r}}_i \,.\delta\mathbf{r}_i - \delta T \tag{4.6}$$

Combining Eqs. (4.3), (4.4) and (4.6)

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{r}}_i \,.\delta\mathbf{r}_i = \delta T + \delta W \tag{4.7}$$

Integrating with respect to time between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \left[\sum_i m_i \dot{\mathbf{r}}_i \,.\delta\mathbf{r}_i\right]_{t_1}^{t_2} \tag{4.8}$$

The right hand side is zero as $\delta \mathrm{r}_i(t_1) = \delta \mathrm{r}_i(t_2) = 0$. Replacing δW by Eq. (4.4), Eq. (4.8) reduces to

$$\int_{t_1}^{t_2} \left(\delta T + \sum_{j=1}^{n} Q_j\, \delta q_j\right) dt = 0 \tag{4.9}$$

Eq. (4.9) is sometimes referred to as *the integral form of D Alembert's principle* or *the generalized version of Hamilton's principle*. The integral form is more advantageous since it is independent of the choice of co-ordinates with which we describe the system. If the external forces are conservative, $\mathbf{F}_i = -\nabla_i V$ and by Eq. (3.33), $Q_j = (-\partial V / \partial q_j)$. Consequently,

$$\sum_j Q_j\, \delta q_j = -\sum_j \frac{\partial V}{\partial q_j} \delta q_j = -\,\delta V \tag{4.10}$$

and Eq. (4.9) becomes

$$\int_{t_1}^{t_2} (\delta T - \delta V) dt = 0$$

$$\int_{t_1}^{t_2} \delta(T - V) dt = \int_{t_1}^{t_2} \delta L\, dt = 0 \tag{4.11}$$

For a holonomic system the *d*-variation and integration can be interchanged.

Then $\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$ (4.12)

which is Hamilton's principle.

4.3 LAGRANGE'S EQUATION FROM HAMILTON'S PRINCIPLE

The action integral must have a stationary value for the actual path. Let us label each possible path in the configuration space by an infinitesimal parameter, say *a*. That is, the set of paths may be labelled by (*q*, *a*) with

q (*t*, 0) representing the correct path. In terms of the parameter *a*, each path may be written as q_i (*t*, *a*) = q_i (*t*, *0*) + *a* h_i (*t*), *i* = 1, 2, 3,..., *n* (4.13) where h_i(t) is a completely arbitrary well-behaved function of time with the condition *h* (t_1) = $h(t_2)$ = 0. From Eq. (4.13)

$$\delta q_i = \eta_i(t)\delta\alpha \quad (4.14)$$

As the q_i's and $\dot{q}_i$'s are only functions of *t* and *a*, for a given h_i(*t*), the action integral *I* is a function of *a* only:

$$I(\alpha) = \int_{t_1}^{t_2} L\{q_i(t, \alpha), \dot{q}_i(t, \alpha), t\} dt \quad (4.15)$$

Expanding the integrand *L* by Taylor series

$$I(\alpha) = \int_{t_1}^{t_2} \sum_i \left[L\{q_i(t,0), \dot{q}_i(t,0), t\} + \frac{\partial L}{\partial q_i}(\alpha\eta_i) + \frac{\partial L}{\partial \dot{q}_i}(\alpha\dot{\eta}_i) \right] dt \quad (4.16)$$

where the higher order terms in the expansion are left out, which is reasonable as *a* 0 . Since the integration limits t_1 and t_2 are not dependent, differentiating with respect to *a* under the integral sign

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} \eta_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i \right] dt \tag{4.17}$$

Integrating the second term on the right hand side by parts

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i dt = \left[\frac{\partial L}{\partial \dot{q}_i} \eta_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta_i(t) \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) dt \tag{4.18}$$

The integrated term vanishes since $\eta_i\,(t_1) = \eta_i(t_2) = 0$. Substituting Eq. (4.18) in Eq. (4.17) we get

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \eta_i(t)\, dt \tag{4.19}$$

Multiplying Eq. (4.19) by $\delta\alpha$

$$\frac{\partial I}{\partial \alpha} \delta\alpha = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta\alpha\, \eta_i(t)\, dt$$

Using Eq. (4.14) and remembering that the left side is simply δI.

$$\delta I = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i(t)\, dt \tag{4.20}$$

For I to be stationary, $dI = 0$. Since q_i's are independent, the variations dq_i's are arbitrary and the necessary condition for the right side of Eq. (4.20) to be zero is that the coefficients of dq_i's vanish separately. Hence,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, \ldots, n \tag{4.21}$$

which is Lagrange's equation, given by Eq. (3.38).

The above result is a special case of the more general Euler-Lagrange differential equation which determines the path $y = y(x)$ such that the line integral $$I = \int_{x_1}^{x_2} f(y, \dot{y}, x)dx \qquad \dot{y} = \frac{dy}{dx} \tag{4.22}$$

has a stationary value. From simple variational considerations, Euler has shown that the necessary and sufficient condition for the integral in Eq. (4.22) to have a stationary value is $$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \tag{4.22a}$$

Later this was applied to mechanical systems by Lagrange. Hence, the name **Euler-Lagrange differential equation** for Eq. (4.22a).

If the forces of the system are not conservative, one has to go back to the generalized version of Hamilton's principle, given by Eq. (4.9).

$$\int_{t_1}^{t_2} \left(\delta T + \sum_i Q_i \, \delta q_i \right) dt = 0$$

The first term on the left is

$$\int_{t_1}^{t_2} \delta T \,(q, \dot{q}, t)\, dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

$$= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial T}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i \right] dt$$

The integral

$$\int_{t_1}^{t_2} \sum_i \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \delta q_i \right) dt = 0$$

since $\delta q_i = 0$ at time t_1 and t_2. Hence,

$$\int_{t_1}^{t_2} \delta T \,(q, \dot{q}, t)\, dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i \, dt \tag{4.23}$$

Combining this with the term $\sum_i Q_i \, \delta q_i$, we have

$$\int_{t_1}^{t_2} \sum_i \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} + Q_i \right) \delta q_i dt = 0 \tag{4.24}$$

Since δq_i 's are arbitrary and independent, it follows from Eq. (4.24)

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \qquad i = 1, 2, 3, \ldots, n \tag{4.25}$$

These equations are the same as Eq. (3.32).

4.4 HAMILTON'S PRINCIPLE FOR NON-HOLONOMIC SYSTEMS

Equations of constraints are in the form of algebraic expressions or in the differential form. If they are in the differential form, they can be directly incorporated into Lagrange's equations by means of Lagrange undetermined multipliers. We first discuss the Lagrange multiplier method for non-holonomic systems and then for holonomic systems.

With non-holonomic system, the generalized co-ordinates are not independent of each other. Consequently, for a virtual displacement consistent with the constraints, the dq_i's are no longer independent. However, a straightforward treatment is possible if the equations of constraints are of the type

$$\sum_j a_{ij}\, dq_j + a_{it} dt = 0 \qquad i = 1, 2, 3, \ldots, m \tag{4.26}$$

where the coefficients a_{ij} and a_{it} may be functions of the q's and time. The quantity m indicates that there are m equations of this type. The constraint equation, Eq. (4.26), for virtual displacement is:

$$\sum_j a_{ij}\, \delta q_j = 0 \qquad i = 1, 2, 3, \ldots, m \tag{4.27}$$

We can now use Eq. (4.27) to reduce the number of virtual displacements to independent ones by the Lagrange multiplier method. If Eq. (4.27) is valid, it is also true that

$$\lambda_i \sum_j a_{ij}\, \delta q_j = 0 \qquad i = 1, 2, 3, \ldots, m \tag{4.28}$$

where $\lambda_i, i = 1, 2, 3, \ldots, m$ are some undetermined quantities known as *Lagrange's multipliers*. In general, they are functions of the co-ordinates and of time t. In addition, Hamilton's principle is assumed to hold for non-holonomic systems. Proceeding as in Section 4.3

$$\delta \int_1^2 L dt = \int_1^2 dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j = 0 \tag{4.29}$$

where the integration is with respect to time t_1 (point 1) to time t_2 (point 2). Summing Eq. (4.28) over i and then integrating with respect to time between points 1 and 2, we have

$$\int_1^2 dt \sum_i \sum_j \lambda_i a_{ij}\, \delta q_j = 0 \tag{4.30}$$

Combining Eqs. (4.29) and (4.30)

$$\int_1^2 dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_i \lambda_i \, a_{ij} \right) \delta q_j = 0 \tag{4.31}$$

In Eq. (4.31) the selection of l_i's are at our disposal but the dq_j's are still not independent but they are connected by the m relations of Eq. (4.27). That is, out of the n co-ordinates $n–m$ may be selected independently and the remaining m are fixed by Eq. (4.27). The integrand in Eq. (4.31) can be split into two as

$$\int_1^2 dt \sum_{j=1}^{n-m} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_i \lambda_i a_{ij} \right) \delta q_j$$

$$+ \int_1^2 dt \sum_{j=n-m+1}^{n} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_i \lambda_i a_{ij} \right) \delta q_j = 0 \tag{4.32}$$

As the values of λ_i's are at our disposal, we now choose them to be such that the second term in Eq. (4.32) vanishes. That is,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_i \lambda_i a_{ij} = 0 \qquad j = (n-m+1), \ldots, n \tag{4.33}$$

Now we are free to select the $n - m$ co-ordinates arbitrarily and therefore from Eq. (4.32)

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_i \lambda_i a_{ij} = 0 \qquad j = 1, 2, \ldots, (n-m) \tag{4.34}$$

Combining Eqs. (4.33) and (4.34), we get the complete set of Lagrange's equations for non-holonomic systems:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_{i=1}^{m} \lambda_1 a_{ij} \qquad j = 1, 2, \ldots, n \tag{4.35}$$

Now we have $n + m$ unknowns, the n co-ordinates (q_j's) and the m Lagrange multipliers (l_i). Eq. (4.35) together with the equations of constraints, now being written as first order differential equations

$$\sum_j a_{ij} \dot{q}_j + a_{it} = 0 \tag{4.36}$$

constitute ($n + m$) equations for $n + m$ unknowns.

Next we consider the physical significance of the undetermined multipliers. For that, let us remove the constraints on the system in such way that the motion is unchanged by the application of external forces Q_j. These external forces Q_j make the equations of motion of the system remain the same. Under the influence of these forces Q_j, the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j \qquad (4.37)$$

which must be identical with Eq. (4.35). Hence, we can identify $\sum_i \ddot{e}_i a_{ij}$ with Q_j, the generalized forces of constraint.

Next we consider a holonomic system in which there are more generalized co-ordinates than degrees of freedom. A holonomic equation of constraint

$$f\left(q_1, q_2, \ldots, q_n, t\right) = 0 \qquad (4.38)$$

is equivalent to the differential equation

$$\sum_j \frac{\partial f}{\partial q_j} dq_j + \frac{\partial f}{\partial t} dt = 0 \qquad (4.39)$$

which is identical in form to Eq. (4.26):

$$a_{ij} = \frac{\partial f}{\partial q_j} \quad \text{and} \quad a_{it} = \frac{\partial f}{\partial t} \qquad (4.40)$$

Thus, Lagrange's multiplier method can be used for holonomic constraints when (i) it is inconvenient to reduce all the q's to independent co-ordinates or (ii) we desire to obtain the forces of constraint.

WORKED EXAMPLES

Example 4.1 Show that the shortest distance between two points is a straight line.

Solution: In a plane, element of arc length

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + (dy/dx)^2}$$

The length of any curve between points 1 and 2 is given by

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\ dx$$

The shortest path can be determined using Euler-Lagrange equation. Writing

$$f = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + \dot{y}^2} \qquad \dot{y} = \frac{dy}{dx}$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

Substituting in Eq. (4.22a)

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0 \qquad \text{or} \qquad \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{constant}$$

Squaring and simplifying

$$\dot{y} = a \qquad a = \text{constant}$$

Integrating

$$y = ax + b \quad b = \text{constant}$$

which is the equation of a straight line.

Example 4.2 The **brachistochrone problem** is to find the curve joining two points along which a particle falling from rest under the influence of gravity reaches the lower point in the least time.

Solution: Let v be the speed along the curve.

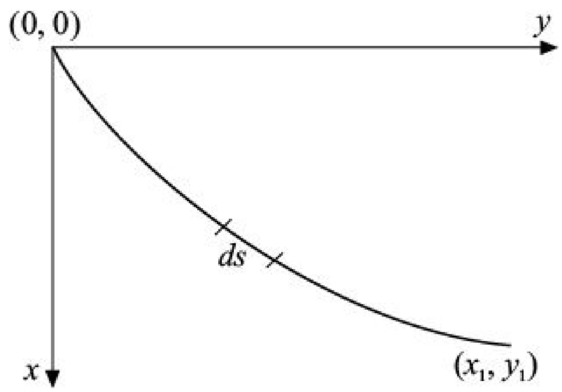


Fig. 4.2 Brachistochrone problem.

The time required to fall an arc length $ds = \frac{ds}{v}$

Time required to reach the lower point $t = \int \frac{ds}{v}$

The problem is to find the minimum of the integral for *t*. If *x* is measured down, by the principle of conservation of energy

$$\frac{1}{2} m v^2 = mgx \qquad \text{or} \qquad v = \sqrt{2gx}$$

The expression for *t* becomes

$$t = \int \frac{ds}{\sqrt{2gx}} = \int \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gx}} dx$$

The function *f* in Euler-Lagrange equation is then

$$f = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gx}}$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{2gx}\,\sqrt{1+\dot{y}^2}} \qquad \frac{\partial f}{\partial y} = 0$$

Substituting in Eq. (4.22a), we get

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{2gx}\,\sqrt{1+\dot{y}^2}}\right) = 0 \qquad \text{or} \qquad \frac{\dot{y}}{\sqrt{2gx}\,\sqrt{1+\dot{y}^2}} = \text{constant}$$

$$\dot{y}^2 = cx(1+\dot{y}^2) \qquad c = \text{constant}$$

$$\dot{y}^2 = \frac{cx}{1-cx} = \frac{x}{b-x} \qquad b = \text{constant}$$

$$\frac{dy}{dx} = \sqrt{\frac{x}{b-x}}$$

Integrating

$$y = \int \sqrt{\frac{x}{b-x}}\, dx$$

The integral can be evaluated making the substitution

$$x = b\sin^2\theta \qquad \text{or} \qquad dx = 2b\sin\theta\cos\theta\,d\theta$$

$$\sqrt{\frac{x}{b-x}}\,dx = \sqrt{\frac{b\sin^2\theta}{b-b\sin^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$= 2b\sin^2\theta\,d\theta = b(1-\cos 2\theta)\,d\theta$$

$$y = b\int(1-\cos 2\theta)\,d\theta = b\left(\theta - \frac{\sin 2\theta}{2}\right) + \text{constant}$$

The constant is zero since the curve passes through (0, 0). Hence,

$$x = b\sin^2\theta = \frac{b}{2}(1-\cos 2\theta) \qquad y = \frac{b}{2}(2\theta - \sin 2\theta)$$

Writing $\phi = 2\theta$ and $\frac{b}{2} = a$, we have

$$x = a(1-\cos\phi) \qquad \text{and} \qquad y = a(\phi - \sin\phi)$$

the parametric equations for a cycloid. The constant a can always be determined as the path goes through the final point (x_1, y_1).

Example 4.3 Using Lagrange's method of undetermined multiplier, find the equation of motion and force of constraint in the case of a simple pendulum.

Solution : The system has two co-ordinates *r* and *q*. (see Fig. 4.3).

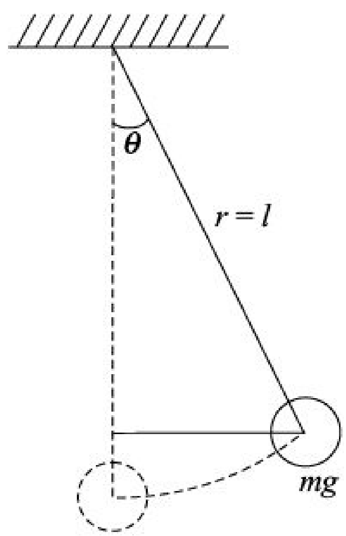


Fig. 4.3 Simple pendulum.

Kinetic energy $T = \frac{1}{2} m r^2 \dot{\theta}^2$

Taking the point of suspension as zero for potential energy,

$$V = -mgr\cos\theta$$

Lagrangian $L = \frac{1}{2} m r^2 \dot{\theta}^2 + mgr\cos\theta$

The only equation of constraint is

$$r = l \qquad \text{or} \qquad dr = 0$$

Comparing this equation with Eq. (4.27), $a_r = 1$ and $a_\theta = 0$.

As there is only one equation of constraint, only one Lagrange multiplier is needed.

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta \qquad \frac{\partial L}{\partial \dot{r}} = 0$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

From Eq. (4.35), the equations of motion are

$$-mr\dot{\theta}^2 - mg\cos\theta = \lambda \tag{i}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0 \tag{ii}$$

From equation (ii), using $r = l$

$$l\ddot{\theta} + g\sin\theta = 0 \tag{iii}$$

When θ is small, $\sin\theta \cong \theta$ and Eq. (iii) reduces to

$$l\ddot{\theta} + g\theta = 0$$

which is the equation of motion of the simple pendulum. Eq. (i) gives the value of λ, which gives the force of constraint, $-ml\dot{\theta}^2 - mg\cos\theta$, where θ = angular displacement. The tension in the string

$$T = -ml\dot{\theta}^2 - mg\cos\theta$$

Example 4.4 Discuss the motion of a disc of mass *m* and radius *b* rolling down an inclined plane without slipping. Also, find the force of constraint using the

Lagrange method of undetermined multipliers.

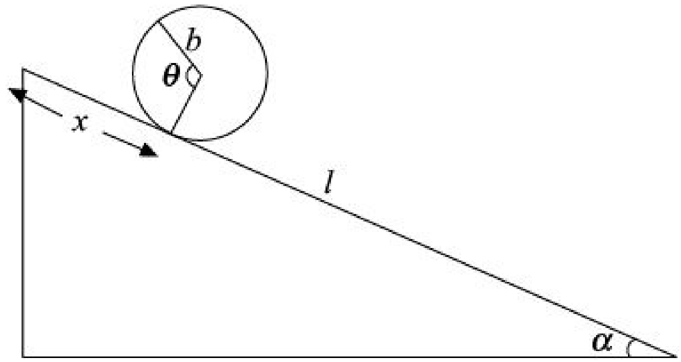


Fig. 4.4 Disc rolling down an incline.

Solution: Figure 4.4 illustrates the disc rolling down the incline. We can take *x* and *q* as the generalized co-ordinates. Moment of inertia of the disc about an axis

axis passing through the centre = $\frac{mb^2}{2}$.

Assuming the potential energy V at the bottom to be zero

$$V = mg\,(l - x)\sin\alpha \tag{i}$$

Total kinetic energy T = Translational K.E. + Rotational K.E.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\,\dot{\theta}^2 \tag{ii}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}mb^2\dot{\theta}^2 - mg(l - x)\sin\alpha \tag{iii}$$

The equation of the constraint (holonomic) connecting the co-ordinates *x* and *q* is

$$x = b\theta \qquad \text{or} \qquad b\theta - x = 0 \tag{iv}$$

The equation of constraint written as first order differential equation is

$$bd\theta - dx = 0$$

Comparing it with Eq. (4.27), the coefficients in the constraint equations are

$$a_x = -1 \qquad \text{and} \qquad a_\theta = b \tag{v}$$

Only one Lagrange multiplier is needed as there is only one equation of constraint.

For x co-ordinate: $\frac{\partial L}{\partial x} = mg \sin\alpha$ $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

For θ co-ordinate: $\frac{\partial L}{\partial \theta} = 0$ $\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mb^2\dot{\theta}$

Lagrange's equations, Eq. (4.35), for x and θ are

$$m\ddot{x} - mg\sin\alpha + \lambda = 0 \quad \text{(vi)}$$

$$\frac{1}{2}mb^2\ddot{\theta} - \lambda b = 0 \quad \text{(vii)}$$

From the constraint equation, we have

$$\ddot{x} = b\ddot{\theta} \quad \text{(viii)}$$

The three equations (vi), (vii) and (viii) may be solved to get the values of $\ddot{x}$, $\ddot{\theta}$ and λ

$$\ddot{x} = \frac{2}{3}g\sin\alpha \quad \ddot{\theta} = \frac{2}{3}\frac{g}{b}\sin\alpha \quad \lambda = \frac{mg}{3}\sin\alpha \quad \text{(ix)}$$

The value of *l* gives the force of constraint resulting from a frictional force. It is this force that reduces the acceleration due to gravity from *g* sin *a* to (2/3) *g* sin *a* when there is friction.

Example 4.5 A bead of mass *m* slides freely on a frictionless circular wire of radius *a* that rotates in a horizontal plane about a point on the circular wire with a constant angular velocity *w* . Find the reaction of the wire on the bead.

Solution: In Example 3.11, we were interested in the equation of motion of the bead. The angle *q* was taken as the generalized co-ordinate as it was a case of one degree of freedom. Now we have to determine the force of constraint too. Hence, the problem is treated as having two degrees of freedom with co-ordinates *r* and *q*. The condition of constraint is *r* = *a* or *dr* = 0 (i)

Hence, $a_r = 1$ and $a_\theta = 0$

The co-ordinates x and y of the bead are given by

$$x = a\cos\omega t + r\cos(\theta + \omega t) \quad \text{(ii)}$$

$$y = a\sin\omega t + r\sin(\theta + \omega t) \quad \text{(iii)}$$

$$\dot{x} = -a\omega\sin\omega t + \dot{r}\cos(\theta + \omega t) - (r\dot{\theta} + r\omega)\sin(\theta + \omega t) \quad \text{(iv)}$$

$$\dot{y} = a\omega\cos\omega t + \dot{r}\sin(\theta + \omega t) + (r\dot{\theta} + r\omega)\cos(\theta + \omega t) \quad \text{(v)}$$

Since the circular wire is in a horizontal plane, potential energy is zero. Hence,

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[a^2\omega^2 + \dot{r}^2 + r^2(\dot{\theta} + \omega)^2 + 2a\omega r(\dot{\theta} + \omega)\cos\theta + 2a\omega\dot{r}\sin\theta\right] \quad \text{(vi)}$$

Substituting the value of L in Eq. (4.35)

$$m[\ddot{r} + a\omega\dot{\theta}\cos\theta - r(\dot{\theta} + \omega)^2 - a\omega(\dot{\theta} + \omega)\cos\theta] = \lambda \quad \text{(vii)}$$

Using the condition $r = a = \text{constant}$

$$\lambda = -ma[(\dot{\theta} + \omega)^2 + \omega^2\cos\theta] \quad \text{(viii)}$$

In the same way for the θ co-ordinate, we have

$$\ddot{\theta} = -\omega^2\sin\theta \quad \text{(ix)}$$

Multiplying both sides by $2\dot{\theta}\,dt$ and integrating

$$\dot{\theta}^2 = 2\omega^2\cos\theta + \text{constant} \quad \text{(x)}$$

The condition $\dot{\theta} = 0$ when $\theta = 0$ gives constant $= -2\omega^2$. Hence,

$$\dot{\theta}^2 = 2\omega^2\cos\theta - 2\omega^2 = 2\omega^2(\cos\theta - 1) \quad \text{(xi)}$$

The value of *l* can be calculated by substituting $\dot{\theta}$ from Eq. (xi) in Eq. (viii). The value of *l* gives the force of constraint, opposite of it is the reaction of the wire on the bead.

$$\text{Reaction} = ma\omega^2\left[3\cos\theta - 1 + 2\sqrt{2}\sqrt{\cos\theta - 1}\right]$$

Example 4.6 A solid sphere of mass *m* and radius *b* rests on top of another fixed sphere of radius *a*. The upper sphere is slightly displaced and it begins to roll down without slipping. By Lagrange's method of undetermined multipliers, find the normal reaction on the upper sphere and the frictional force at the point of contact.

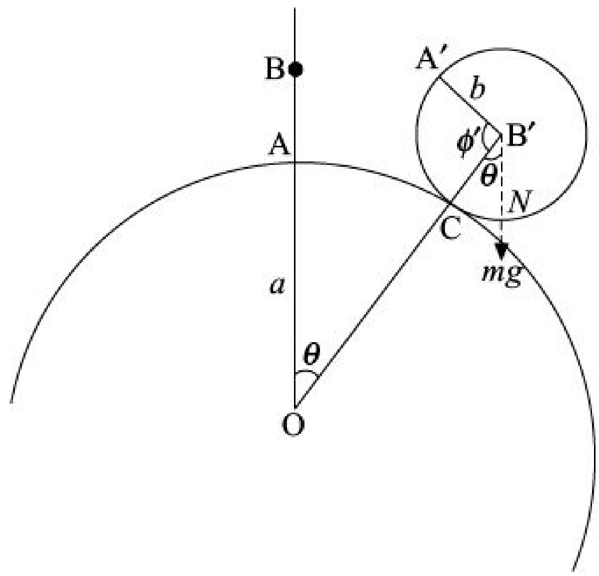


Fig. 4.5 Instantaneous position of a sphere rolling down another sphere.

Solution: Initially the sphere is at the top with the point of contact at A and the centre of the sphere B is along the line OA. When the sphere moves to a position with angle *q* as in Fig. 4.5, the line BA in the body of the sphere takes the position *BA*. In the process the point of contact travels arc AC = arc *CA*. That is

$$a\theta = b\phi' \qquad \text{(i)}$$

We must measure angles with respect to a fixed direction. The angle travelled with respect to the vertical $= \phi' + \theta = \phi.$ The corresponding distance is

$$\text{arc AC} + \text{arc CN} = a\theta + b\theta = (a+b)\theta = r\theta$$

$$r = a + b \qquad \text{(ii)}$$

which is one equation of constraint. This arc length is the distance travelled by the centre B

$$= (a+b)\,\theta = a\theta + b\theta = b\phi' + b\theta = b\,(\theta + \phi') = b\phi$$

The second equation of constraint is:

$$(a+b)\theta = b\phi \qquad \phi = \phi' + \theta \qquad \text{(iii)}$$

Two Lagrange multipliers λ_1 and λ_2 are needed as there are two equations of constraints. In the differential form the constraint equations are

$$dr = 0 \quad \text{and} \quad (a+b)d\theta - bd\phi = 0 \qquad \text{(iv)}$$

Comparing these conditions with Eq. (4.26), we have

$$a_r = 1 \qquad a_\theta = (a+b) \quad \text{and} \quad a_\phi = -b \qquad \text{(v)}$$

The co-ordinates (*r*, *q*, *f*) can be selected as generalized co-ordinates. The kinetic energy (*T*) of the sphere = translational kinetic energy + rotational kinetic energy

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{5}mb^2\dot{\phi}^2 \qquad \text{(vi)}$$

where the second term is the rotational kinetic energy $= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2}{5}mb^2\dot{\phi}^2$.

With respect to O, potential energy $V = mgr\cos\theta$. Then the Lagrangian of the system

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{5}mb^2\dot{\phi}^2 - mgr\cos\theta \qquad \text{(vii)}$$

Lagrange's equations, Eq. (4.35), are then

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda_1 \qquad \text{(viii)}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mgr\sin\theta = (a+b)\lambda_2 \qquad \text{(ix)}$$

$$\frac{2}{5}mb^2\ddot{\phi} = -b\lambda_2 \qquad \text{(x)}$$

In addition to these three equations of motion, we have the two constraint equations, Eqs. (ii) and (iii), to solve for the various quantities. Since $r = (a + b)$ = constant, $\ddot{r} = 0.$ Then Eq. (viii) reduces to

$$m(a+b)\dot{\theta}^2 + mg\cos\theta = \lambda_1 \qquad \text{(xi)}$$

From Eq. (ix)

$$m(a+b)^2\ddot{\theta} - mg(a+b)\sin\theta = (a+b)\lambda_2 \qquad \text{(xii)}$$

From Eqs. (x) and (iii)

$$\lambda_2 = -\frac{2}{5}mb\ddot{\phi} = -\frac{2}{5}m(a+b)\ddot{\theta} \qquad \text{(xiii)}$$

Combining Eqs. (xii) and (xiii)

$$m(a+b)^2\ddot{\theta} - mg(a+b)\sin\theta = (a+b)\left[-\frac{2}{5}m(a+b)\ddot{\theta}\right]$$

$$\ddot{\theta} = \frac{5g\sin\theta}{7(a+b)} \qquad \text{(xiv)}$$

Multiplying by $2\dot{\theta}dt$ and integrating

$$\dot{\theta}^2 = -\frac{10g\cos\theta}{7(a+b)} + \text{constant}$$

Use of the condition that $\dot{\theta} = 0$ when $\theta = 0$ reduces it to

$$\dot{\theta}^2 = \frac{10g}{7(a+b)}(1-\cos\theta) \qquad \text{(xv)}$$

With this value of $\dot{\theta}^2,$ Eq. (xi) reduces to

$$\lambda_1 = \frac{mg}{7}(17\cos\theta - 10) \qquad \text{(xvi)}$$

which represents the reaction of the fixed sphere on the rolling one. It is in the direction of $\hat{e}_r$.

From Eqs. (xiii) and (xiv)

$$\lambda_2 = -\frac{2}{7}mg\sin\theta \qquad \text{(xvii)}$$

The frictional force will be in a direction opposite to the direction of increasing θ.

Frictional force $= \frac{2}{7}mg\sin\theta$

Example 4.7 A particle of mass *m* is placed at the top of a smooth hemisphere of radius *a*. Find the reaction of the hemisphere on the particle. If the particle is disturbed, at what height does it leave the hemisphere?

Solution: Let the two generalized co-ordinates be r and *q* (ses Fig. 4.6). The bottom of the hemisphere is taken as the reference level for potential energy. The kinetic (*T*) and potential (*V*) energies are

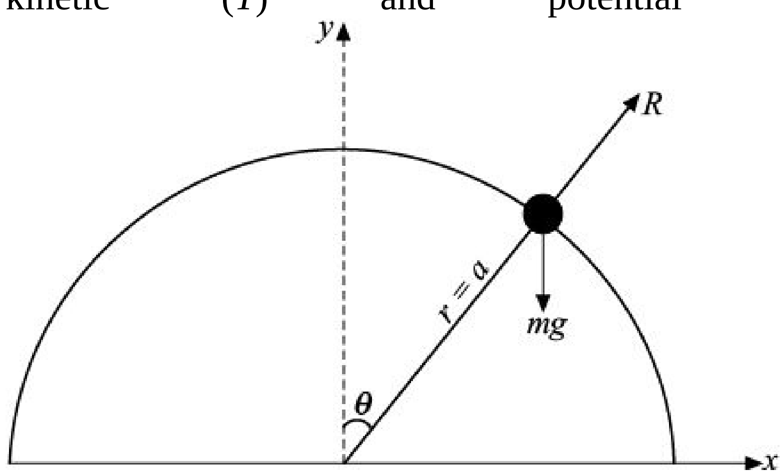


Fig. 4.6 Particle on a hemisphere.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = mgr\cos\theta$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \qquad \text{(i)}$$

The equation of constraint is $r = a$ or $dr = 0$ and therefore $a_r = 1$ and $a_\theta = 0$. Lagrange's equations are

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \qquad \text{(ii)}$$

$$ma^2\ddot{\theta} - mga\sin\theta = 0 \qquad \text{(iii)}$$

Multiplying Eq. (iii) by $2\dot{\theta}dt$ and integrating

$$2\dot{\theta}\,\ddot{\theta}\,dt = \frac{2g}{a}\sin\theta\,dt$$

$$\dot{\theta}^2 = \frac{2g}{a}(1-\cos\theta) \qquad \text{(iv)}$$

where the integration constant is $2g/a$, since $\dot{\theta} = 0$ when $\theta = 0$. Substituting this value of $\dot{\theta}^2$ in Eq. (ii)

$$\lambda = mg(3\cos\theta - 2) \qquad \text{(v)}$$

which is the reaction R. The particle leaves the hemisphere when $\lambda = 0$. That is,

$$3\cos\theta - 2 = 0 \qquad \text{or} \qquad \theta = \cos^{-1}(2/3)$$

REVIEW QUESTIONS

1. State and explain Hamilton's principle, bringing out clearly the nature of variation involved.
2. Express D' Alembert's principle in the integral form. What is its advantage over the one in the differential form?
3. Briefly outline Lagrange's multiplier method for a non-holonomic system.
4. Explain the significance of the Lagrange multiplier constant.
5. Lagrange's multiplier method can be used also for systems having holonomic constraints. How?
6. For a particular set of equations of motion, there is no unique choice of Lagrangian. Comment.
7. State and explain the Euler-Lagrange differential equation in the calculus of variation.

PROBLEMS

1. Obtain the equation of motion of a spring mass system, using Hamilton's variational principle.

2. Consider a curve passing through two fixed end-points (x_1, y_1) and (x_2, y_2) and revolve it about the *y*-axis to form a surface of evolution. Find the equation of the curve for which the surface area is minimum.
3. Consider the motion of a hoop (ring) of mass *m* and radius *r* rolling down without slipping on an inclined plane of length *l* and angle *a*. Obtain Lagrange's equations of motion and hence show that the friction force of constraint is $mg^{\sin \acute{a} / 2}$. Also evaluate the velocity of the hoop at the bottom of the incline.
4. A cylinder of mass *m* and radius *r* is rolling down an inclined plane of length *l* and angle a. Calculate the force of constraint and the velocity of the cyclinder at the bottom of the incline.
5. If *L* is the Lagrangian for a system of *n* degrees of freedom satisfying Hamilton's variational principle, show that $L' = L + \frac{dF(q_1, q_2, \ldots, q_n, t)}{dt}$

also satisfies Hamilton's principle where F is any arbitrary well-behaved function.
6. A cylinder of radius *a* is fixed on its side, and a ring of mass *m* and radius *b* rolls without slipping on it. The ring starts from rest from the top of the cylinder. Using Lagrange's multiplier method, find (i) the reaction on the ring due to the cylinder, and (ii) the position *q* when the ring leaves the cylinder.
7. For identical particles obeying Fermi-Dirac statistics, the probability that n_i particles are in the i^{th} state of energy $\in_i, i = 1, 2, 3, \ldots$ is given by

$$W = \prod_{i=1}^{\infty} \frac{g_i!}{n_i!(g_i - n_i)!}, \quad g_i - \text{degeneracy of the state}$$

where the symbol $\prod_i$ indicates the product of a series of similar terms. Use Lagrange's method of undetermined multipliers to maximize *W* subjects to the conditions (i) $\sum_i n_i = N = \text{constant}$ and (ii) $\sum_i n_i \in_i = E = \text{constant}$

and derive an expression for the most probable distribution of *N* particles among the various states. Assume n_i's, g_i's and *N* to be very large.

[Hint: Use the Stirling approximation, which states that is large.]

8. A particle of mass m is placed at the top of a vertical hoop of radius a. Calculate the reaction of the hoop on the particle by Lagrange multiplier method. Also find the point at which the particle falls off.

5

Central Force Motion

A central force is a force whose line of action is always directed towards a fixed point, called the *centre* or *origin* of the force, and whose magnitude depends only on the distance from the centre. If interaction between any two objects is represented by a central force, then the force is directed along the line joining the centres of the two objects. Central forces are important because we encounter them very often in physics. The familiar gravitational force is a central force. The electrostatic force between two charges is a central force. Even certain two-body nuclear interactions such as the scattering of *a*-particles by nuclei is governed by a central force. In this chapter, we shall discuss some of the salient features of central force motion.

5.1 REDUCTION TO ONE-BODY PROBLEM

Consider an isolated system consisting of two particles of masses m_1 and m_2 with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ as shown in Fig. 5.1. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be their position vectors with respect to the centre of mass (CM) and $\mathbf{R}$ be the position vector of the centre of mass. From the figure we see that

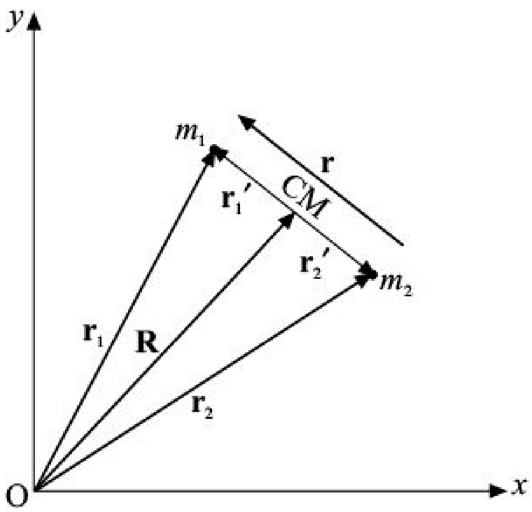


Fig. 5.1 Co-ordinates of the two-body system.

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2' \tag{5.1}$$

and

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_1' \quad \mathbf{r}_2 = \mathbf{R} + \mathbf{r}_2' \tag{5.2}$$

Such a system has six degrees of freedom and hence six generalized co-ordinates are required to describe its motion. The three components of the difference vector **r** and the three components of the vector **R** can be taken as the generalized co-ordinates. By the definition of centre of mass

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \qquad \text{and} \qquad m_1\mathbf{r}_1{}' + m_2\mathbf{r}_2{}' = 0 \tag{5.3}$$

From the second equation of Eq. (5.3)

$$r_2{}' = -\frac{m_1 r_1{}'}{m_2} \tag{5.4}$$

Combining Eqs. (5.1) and (5.4)

$$\mathbf{r} = \mathbf{r}_1{}' - \mathbf{r}_2{}' = \mathbf{r}_1{}' + \frac{m_1\mathbf{r}_1{}'}{m_2} = \frac{(m_1 + m_2)r_1{}'}{m_2}$$

$$\mathbf{r}_1{}' = \frac{m_2\mathbf{r}}{m_1 + m_2} \tag{5.5}$$

By similar arguments

$$\mathbf{r}_2{}' = -\frac{m_1\mathbf{r}}{m_1 + m_2} \tag{5.6}$$

From Eqs. (5.2), (5.5) and (5.6) we get

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2\mathbf{r}}{m_1 + m_2} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_1\mathbf{r}}{m_1 + m_2} \tag{5.7}$$

We shall limit ourselves to cases where the forces acting are directed along the line joining the masses. The Lagrangian of the system can be written as

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(\mathbf{r}_1 - \mathbf{r}_2) \tag{5.8}$$

$$= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2\dot{\mathbf{r}}}{m_1 + m_2}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1\dot{\mathbf{r}}}{m_1 + m_2}\right)^2 - V(\mathbf{r})$$

$$= \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2^2\dot{\mathbf{r}}^2}{(m_1 + m_2)^2} + \frac{1}{2}\frac{m_2 m_1^2\dot{\mathbf{r}}^2}{(m_1 + m_2)^2} - V(\mathbf{r})$$

$$= \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{(m_1 + m_2)^2}(m_2 + m_1)\dot{\mathbf{r}}^2 - V(\mathbf{r})$$

$$M = m_1 + m_2 \quad \text{and} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{5.9}$$

Writing

$$L = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(\mathbf{r}) \tag{5.10}$$

where *m* is called the **reduced mass** of the system. Thus, the central force motion of two bodies splits as a uniform centre of mass motion plus the relative motion of a particle of mass *m* with a relative co-ordinate **r**.

The three components of **R** do not appear in the Lagrangian and therefore they are cyclic. That is, the centre of mass is either at rest or moving at a constant velocity, and we can drop the first term from the Lagrangian in our discussion. The effective Lagrangian *L* is now given by

$$L = \frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(\mathbf{r}) \tag{5.11}$$

It is the Lagrangian of a particle of mass *m* moving in a central force field which is derivable from the potential function *V*(**r**). The problem of two bodies moving under the influence of a mutual central force is thus equivalent to a one-body problem moving about a fixed force centre. Once we have found **r**(*t*), we can calculate $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ from Eq. (5.7).

5.2 GENERAL PROPERTIES OF CENTRAL FORCE

MOTION

The equation of motion for a particle of mass *m* in central force field is

$$\mu \ddot{\mathbf{r}} = \mathbf{F}(r) = F(r)\, \hat{r} = F(r) \frac{\mathbf{r}}{r} \tag{5.12}$$

where $\mathbf{F} = -\partial V / \partial r$. Since it is a vector equation, in effect we have 3 equations. We can learn a lot about the motion of the particle without actually solving these equations.

Angular Momentum

Taking the cross product of both sides of Eq. (5.12) with **r**, we have

$$\mathbf{r} \times \mu \ddot{\mathbf{r}} = \mathbf{r} \times F\,(\mathbf{r})\frac{\mathbf{r}}{r} = \frac{F(\mathbf{r})}{r}\mathbf{r} \times \mathbf{r} = 0 \tag{5.13}$$

Differentiating $\mathbf{L} = \mathbf{r} \times \mu\dot{\mathbf{r}}$ with respect to time

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mu\dot{\mathbf{r}}) = \mu\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \mu\ddot{\mathbf{r}} = 0 \tag{5.14}$$

where we have used Eq. (5.13). It follows from Eq. (5.14) that

L = Constant (5.15) That is, the angular momentum ***L*** of a body under the action of a central force is conserved.

Next consider the dot product of **L** with **r**

$$\mathbf{L} \cdot \mathbf{r} = (\mathbf{r} \times \mu\dot{\mathbf{r}}) \cdot \mathbf{r} = (\mathbf{r} \times \mathbf{r}) \cdot \mu\dot{\mathbf{r}} = 0 \tag{5.16}$$

It means that the angular momentum **L** is normal to the vector **r**. In other words, throughout the motion, the radius vector **r** of the particle lies in a plane perpendicular to the angular momentum. That is, the motion is confined to a plane which is perpendicular to **L**. Thus, the problem has been simplified to a motion in two dimensions instead of three dimensions.

The Lagrangian L can now be expressed in plane polar co-ordinates as

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(\mathbf{r}) \tag{5.17}$$

We see that θ is a cyclic co-ordinate and therefore the corresponding generalized momentum p_θ must be a constant

$$\mathbf{p}_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = l \tag{5.18}$$

where l is a constant. It is a **first integral of motion**. It can be seen from Fig. 5.2(a) that the constant l is simply the magnitude of angular momentum (**L**).

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mu[\mathbf{r} \times \mathbf{v}] = \mu\,[\mathbf{r} \times (\mathbf{v}_r + \mathbf{v}_\theta)]$$
$$= \mu(\mathbf{r} \times \mathbf{v}_r) + \mu\,(\mathbf{r} \times \mathbf{v}_\theta) = \mu\,(\mathbf{r} \times \mathbf{v}_\theta) \tag{5.19}$$

since **r** and $\mathbf{v}_r$ are collinear vectors.

$$|\mathbf{L}| = \mu r\left(r\dot{\theta}\right) = \mu r^2 \dot{\theta} = l \tag{5.20}$$

Law of Equal Areas

The result in Eq. (5.18) has an important consequence. Consider a mass *m* at a distance $\mathbf{r}(q)$ at time *t* from the force centre O as shown in Fig. 5.2(b). In a time interval *dt* the mass moves from *A* to *B*. The distance of *B* from the force centre O is $\mathbf{r}(q + dq)$. As shown in the figure, the radius vector **r** sweeps out an area *dA* in a time *dt*. Since *dq* is very small, *ds* will be small and almost a straight line.

$$dA = \frac{1}{2} r \times r\,d\theta = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \qquad (5.21)$$

which is the rate at which the radius vector sweeps out the area. Substituting the value of $\dot{\theta} = L/mr^2$ in Eq. (5.21)

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant} \qquad (5.22)$$

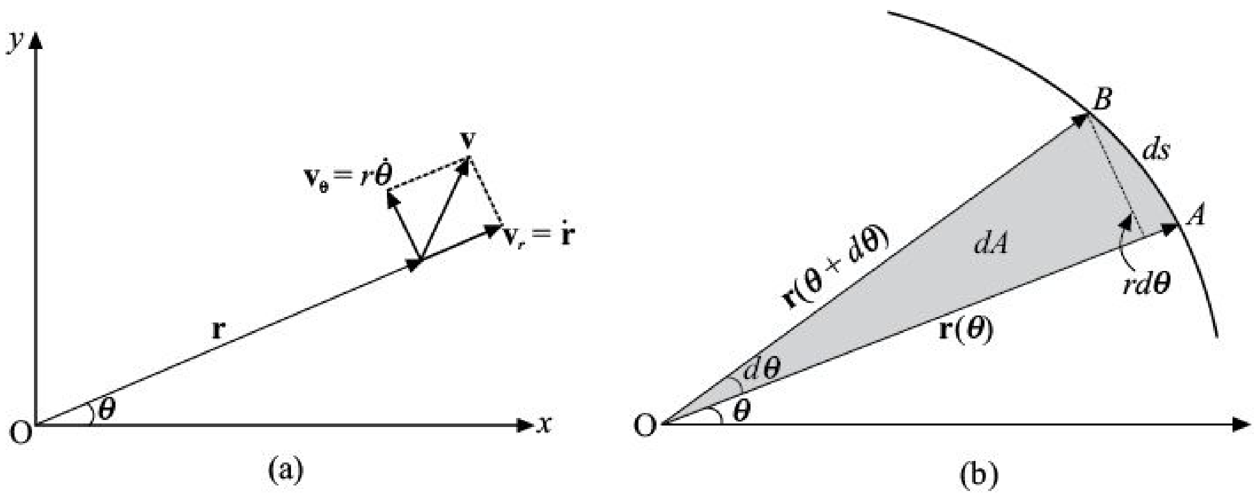


Fig. 5.2 (a) Motion of a particle in plane polar co-ordinates; (b) Area swept by a radius vector **r** in time *dt*.

The conservation of angular momentum implies that the radius vector **r** traces equal areas in equal intervals of time. Eq. (5.22) is a statement of **Kepler's second law** of planetary motion. It is also known as the **law of equal areas**. The law of equal areas is a very general result for any type of central force. Kepler's second law implies that a planet will move faster at a point closer to the sun than at a point farther from it. That is, as *r* increases, velocity decreases to keep the areal velocity constant, which is illustrated in Fig. 5.3. When the motion is periodic with period T, we may integrate Eq. (5.22) and get

$$A = \int dA = \int_0^T \frac{L}{2\mu} dt = \frac{L}{2\mu} T \tag{5.23}$$

The central forces are conserved. Therefore, the total energy of the system is also conserved.

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{constant}$$

It is another first integral of motion. From Eq. (5.20), $\dot{\theta} = L/\mu r^2$. Hence,

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \tag{5.24}$$

Note that the expression for E does not contain $\dot{\theta}$.

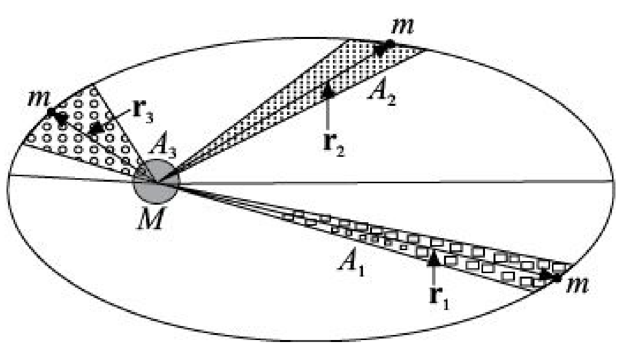


Fig. 5.3 Law of equal areas: $A_1 = A_2 = A_3$.

5.3 EFFECTIVE POTENTIAL

Equation (5.24) is identical to the total energy expression of a particle of mass *m* moving under the influence of an effective potential

$$V_{eff} = V(r) + \frac{L^2}{2\mu r^2} \tag{5.25}$$

The effective potential is the sum of the real potential $V(r)$ and an additional term called the **centrifugal potential** defined as

$$V_c = \frac{L^2}{2\mu r^2} \tag{5.26}$$

This V_{eff} is equivalent to a force term

$$F_{eff}(r) = F(r) + \frac{L^2}{\mu r^3} \tag{5.27}$$

The additional force term $L^2/\mu\, r^3$, a fictitious force, is usually referred to as the centrifugal force, F_c:

$$F_c = \frac{L^2}{\mu r^3} = \mu r \dot{\theta}^2 \tag{5.28}$$

Since L is constant, it is evident that F_c is also a central force, with this effective force the equation of motion is identical to that of a one-dimensional one:

$$\mu \ddot{r} = F_{eff} \tag{5.29}$$

In physical terms, Eq. (5.29) describes the motion of a particle of mass *m* under a central force as viewed by an observer at the centre of force from a rotating reference frame. In this rotating frame the force seems to be $F_{eff}(r)$.

5.4 CLASSIFICATION OF ORBITS

A system has two generalized co-ordinates *r* and *q* and therefore two second order differential equations have to be solved to study the motion. In other words, four integrations are needed to study the motion of a particle. However, without solving the equations of motion for a specific central force, we can learn a lot about the motion using the first integrals of motion, *E* and *L*. Solving Eq. (5.24) for $\dot{r}$ we have

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)} \tag{5.30}$$

Since $\dot{r}$ must be positive

$$\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right) \geq 0 \quad \text{or} \quad E \geq V(r) + \frac{L^2}{2\mu r^2} \tag{5.31}$$

When the equality sign in Eq. (5.31) is valid, $\dot{r} = 0$ which corresponds to the maximum and minimum values of *r*.

To get further insight, we consider motion in an attractive inverse square force field :

$$F(r) = -\frac{k}{r^2} \qquad V(r) = -\frac{k}{r} \qquad k > 0 \tag{5.32}$$

Then, V_{eff} becomes

$$V_{eff} = -\frac{k}{r} + \frac{L^2}{2\mu r^2} \tag{5.33}$$

For large *r*, the first term is the dominant one and therefore $V_{eff} < 0$. As *r* , V_{eff} 0. For small *r*, the second term is the dominant one and V_{eff} + as *r* 0. Fig. 5.4 shows a plot of $V_{eff}(r)$ versus *r* for a particular value of *L*.

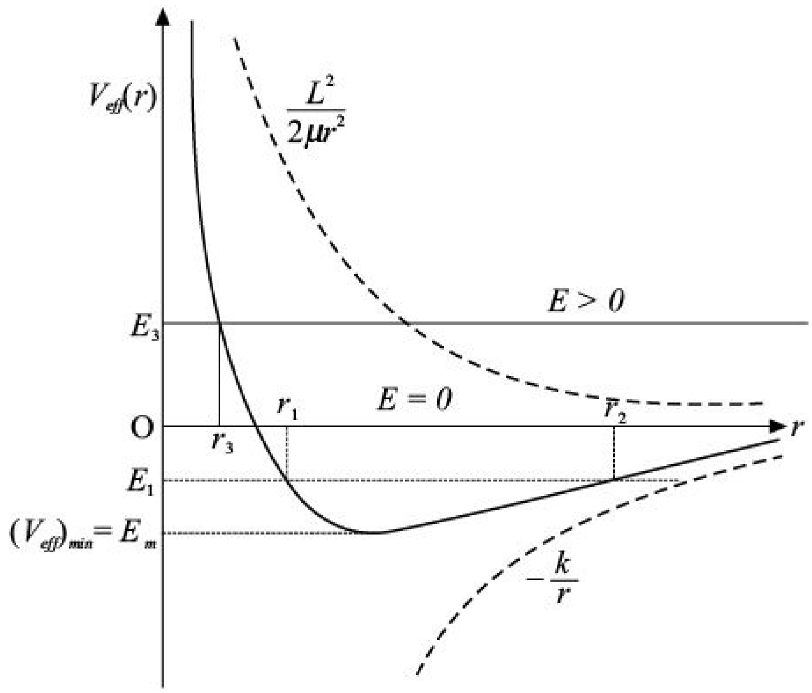


Fig. 5.4 Plot of $V_{eff}(r)$ versus r for a given L for attractive inverse square law force.

Next let us consider motions for different values of E. We have four distinct cases.

E > 0: If $E > 0$, say E_3, from Fig. 5.4 it is evident that there is a minimum radial distance r_3 but no maximum. The motion of the particle is unbounded. A particle heading towards the centre of force can come as close as r_3 and then turns back and may even go back to infinity. Thus, for a particle with $E > 0$, the motion is unbounded with a single turning point.

$0 > E > (V_{eff})_{min} = E_m$: This condition corresponds to energy E_1, in Fig. 5.4. The radial motion of the particle will be confined to the values of $r = r_1 = r_{min}$ and

$r = r_2 = r_{max}$. The points r_1 and r_2 are the turning points. At these points

$$E = -\frac{k}{r} + \frac{L^2}{2\mu r^2} \qquad (5.34)$$

Though $\dot{r} = 0$ at these points, angular velocity $\dot{\theta} \neq 0$. Hence, the particle will not be at rest at these points. Actually, the motion is confined between the areas of two circles of radii r_1 and r_2. A possible shape of the path for an attractive inverse square law force is an ellipse with the focus at the force centre. When r

varies from r_1 to r_2 and back, the radius vector turns through an angle *q* which can easily be obtained from Eq. (5.20):

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \quad \text{or} \quad \theta = \int \frac{L}{\mu r^2} dt + \text{constant} \tag{5.35}$$

From Eq. (5.30)

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)}} \tag{5.36}$$

Substituting this value of *dt* in Eq. (5.35), we get

$$\theta = \int \frac{\left(L/r^2\right) dr}{\sqrt{2\mu\left(E - V\right) - L^2/r^2}} + \text{constant} \tag{5.37}$$

When the angle *q* = 2*p* (*m/n*) where *m* and *n* are integers, the path is a closed orbit. That is, during *n* periods the radius vector of the particle makes *m* complete revolutions and will come back to its original position. When *q* is not a rational fraction of 2*p*, the path has the shape of a **rosette**, as shown in Fig. 5.5. Such an orbital motion is often referred to as a **precessing motion**.

$E = E_m = (V_{eff})_{min}$: If the energy of the particle is such that $E = (V_{eff})_{min}$, $\dot{r} = 0$ and $\dot{\theta}$ is finite. Hence, the particle must move in a circle.

$E < (V_{eff})_{min}$: If the energy of the particle is less than $(V_{eff})_{min}$, $\dot{r}$ will be imaginary and therefore no physically meaningful motion is possible.

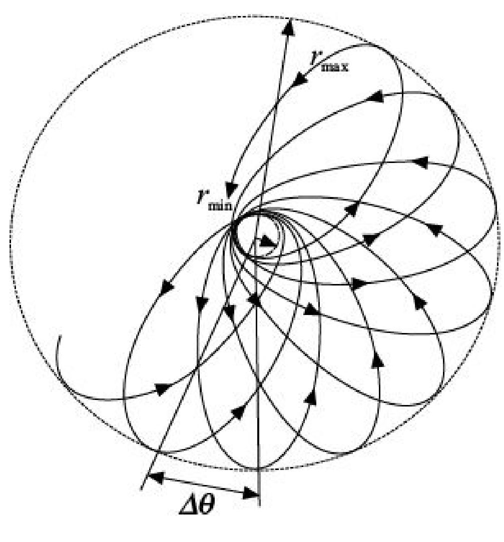


Fig. 5.5 Motion of a particle with energy $0 > E > (V_{eff})_{min}$ resulting in precessing motion.

5.5 MOTION IN A CENTRAL FORCE FIELD—GENERAL SOLUTION

The complete solution for the motion of a particle in a central force field can be obtained in two ways: the energy method and Lagrangian analysis. The energy method is based on the laws of conservation of energy and angular momentum. In Lagrangian analysis, the differential equations of motion are obtained from the Lagrangian and then analysed.

Energy Method

In Eq. (5.24) we have the energy expression in terms of the angular momentum. Solving it for r we get

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)} \tag{5.38}$$

Integrating

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)}} \tag{5.39}$$

which gives the general solution in the form $t = t(r)$. The solution in the standard form $\boldsymbol{r} = \boldsymbol{r}(t)$ is also possible from Eq. (5.38). Often we require a relation between q and r, the form of which is given in Eq. (5.37). The integral in Eq. (5.37) can be put in a standard form by making the substitution

$$u = \frac{1}{r} \quad \text{or} \quad du = -\frac{dr}{r^2} \tag{5.40}$$

Equation (5.37) now takes the form

$$\theta = \theta_0 - L\int \frac{du}{\sqrt{2\mu(E - V) - L^2 u^2}} \tag{5.41}$$

where q_0 is the constant of integration. To proceed further, we require the form of the potential $V(r)$.

Lagrangian Analysis

The Lagrangian of the system is given by Eq. (5.17). Lagrange's equations of motion are

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad (5.42)$$

$$\frac{d}{dt}\left(\mu r^2\dot{\theta}\right) = 0 \qquad (5.43)$$

Replacing $-\partial V/\partial r$ by $F(r)$

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = F(r) \qquad (5.44)$$

It is convenient to work with the new variable $u = \frac{1}{r}$. In terms of u

$$\dot{\theta} = \frac{L}{\mu r^2} = \frac{L}{\mu}u^2 \qquad (5.45)$$

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta}$$

$$= -\frac{L}{\mu}\frac{du}{d\theta} \qquad (5.46)$$

$$\frac{d^2r}{dt^2} = -\frac{L}{\mu}\frac{d}{dt}\frac{du}{d\theta} = -\frac{L}{\mu}\frac{d^2u}{d\theta^2}\dot{\theta} = -\left(\frac{Lu}{\mu}\right)^2\frac{d^2u}{d\theta^2} \qquad (5.47)$$

Substituting Eqs. (5.47) and (5.45) in Eq. (5.44), we get

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{L^2u^2}F\left(\frac{1}{u}\right) \qquad (5.48)$$

which is the differential equation of the orbit. It is possible to find the force law if the equation of the orbit **r** = **r** (*q*) is given.

5.6 INVERSE SQUARE LAW FORCE

The most important type of central force is the one in which the force varies inversely as the square of the radial distance:

$$V(r) = -\frac{k}{r} \qquad \text{or} \qquad F(r) = -\frac{k}{r^2} \qquad (5.49)$$

where k is a positive constant for an attractive force and negative for a repulsive force. The two most important cases under this category are gravitational force and coulomb force. For the gravitational force $k = G\ m_1\ m_2$ where G is the gravitational constant. The equation of the orbit can be obtained from Eq. (5.41) which now takes the form

$$\theta = \theta_0 - \int \frac{du}{[(2\mu E/L^2) + (2\mu k/L^2)u - u^2]^{1/2}} \tag{5.50}$$

The integral on the right side of Eq. (5.50) is a standard one of the type

$$\int \frac{dx}{(a + bx + cx^2)^{1/2}} = \frac{1}{\sqrt{-c}} \cos^{-1}\left(-\frac{b + 2cx}{(b^2 - 4ac)^{1/2}}\right) \tag{5.51}$$

To apply this standard integral to Eq. (5.50) we have to set

$$a = \frac{2\mu E}{L^2} \qquad b = \frac{2\mu k}{L^2} \qquad c = -1$$

$$(b^2 - 4ac)^{1/2} = \left(\frac{4\mu^2 k^2}{L^4} + \frac{8\mu E}{L^2}\right)^{1/2} = \frac{2\mu k}{L^2}\left(1 + \frac{2EL^2}{\mu k^2}\right)^{1/2}$$

with these values, the integral in Eq. (5.50) reduces to

$$\int \frac{du}{[(2\mu E/L^2) + (2\mu k/L^2)u - u^2]^{1/2}} = \cos^{-1} \frac{(uL^2/\mu k) - 1}{[1 + (2EL^2/\mu k^2)]^{1/2}}$$

Equation (5.50) now takes the form

$$\cos(\theta - \theta_0) = \frac{(uL^2/\mu k) - 1}{[1 + (2EL^2/\mu k^2)]^{1/2}}$$

$$u = \frac{1}{r} = \frac{\mu k}{L^2}\left[1 + \sqrt{1 + \frac{2EL^2}{\mu k^2}} \cos(\theta - \theta_0)\right] \tag{5.52}$$

which is the equation of the orbit. It may be noted that only three (q_0, E and L) of the four constants of integration appear in the orbit equation. The fourth constant can be obtained by finding the solution of the other equation of motion, Eq. (5.43).

The general equation of a conic with one focus at the origin is

$$\frac{1}{r} = C\left[1 + \epsilon\cos(\theta - \theta_0)\right] \quad (5.53)$$

where is the eccentricity of the conic section. A comparison of Eqs. (5.52) and (5.53) shows that the orbit is always a conic section with eccentricity

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}} \quad (5.54)$$

$$C = \frac{\mu k}{L^2} \quad (5.55)$$

The nature of the orbit depends on the value of ϵ according to the following scheme:

$\epsilon > 1$	$E > 0$	hyperbola
$\epsilon = 1$	$E = 0$	parabola
$0 < \epsilon < 1$	$E < 0$	ellipse
$\epsilon = 0$	$E = -\dfrac{\mu k^2}{2L^2}$	circle

These orbits are shown in Fig. 5.6. It may be noted that the energy is negative for bound orbits.

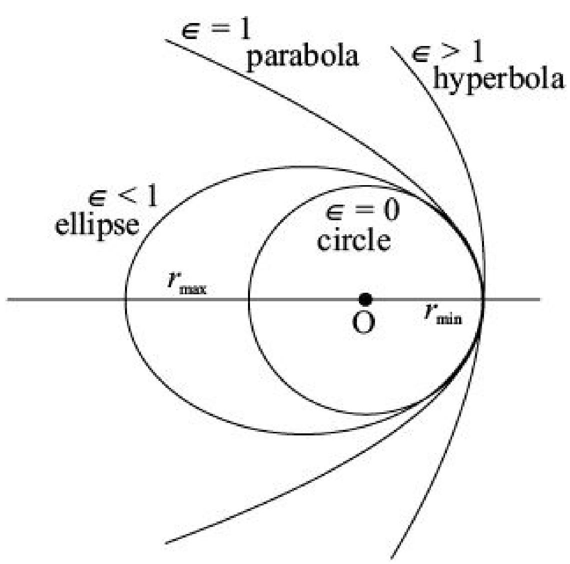


Fig. 5.6 Shapes of different conics.

The constant q_0 simply determines the orientation of the orbit. Therefore, we can even select $q_0 = 0$ which corresponds to measuring q from r_{min}. Then the equation of the orbit, Eq. (5.53), becomes

$$\frac{1}{r} = C(1 + \epsilon \cos\theta) \tag{5.56}$$

The position corresponding to $r = r_{min}$ is called the **pericentre** whereas that corresponding to r_{max} is the **apocentre**. For motion about the sun, the corresponding positions are **perihelion** and **aphelion** and for motion about the earth they are **perigee** and **apogee.** The general term for the turning points is **apsides.**

5.7 KEPLER'S LAWS

Based on the detailed astronomical data of Tycho Brahe, Kepler enunciated three general laws regarding planetary motion. They can be stated as follows: **Law of orbits**: Planets move in elliptical orbits with the sun at one focus.

Law of areas: The radius vector from the sun to a planet sweeps equal areas in equal intervals of time.

Law of periods: The square of the period of revolution about the sun is proportional to the cube of the semi-major axis of its orbit.

We have already discussed the first two laws in Sections 5.2 and 5.6. The laws of orbits follows directly from Newton's law of gravitation, that is, from the inverse square nature of the force of gravitation. The law of areas is a consequence of the result that the angular momentum remains constant. In this section we consider the third law in detail.

In the case of ellipse, the perihelion ($r_1 = r_{min}$) and aphelion ($r_2 = r_{max}$) distances are the values of r when $(q - q_0) = 0$ and p, respectively. Now from Eq. (5.53)

$$r_{min} = r_1 = \frac{1}{C(1+\epsilon)} \quad \text{and} \quad r_{max} = r_2 = \frac{1}{C(1-\epsilon)} \tag{5.57}$$

Semi-major axis

$$a = \frac{r_1 + r_2}{2} = \frac{1}{C(1-\epsilon^2)} \tag{5.58}$$

Substituting the values of C and ϵ from Eqs. (5.55) and (5.54)

$$a = -\frac{k}{2E} \tag{5.59}$$

That is, the length of the semi-major axis depends solely on the energy.

Let *T* be the time period of an elliptical orbit. From Eq. (5.23), the area of the ellipse $A = LT/2m$. The area of the ellipse is also equal to *pab*, where *b* is the length of the semi-minor axis. From these two relations, we have

$$\frac{LT}{2\mu} = \pi ab$$

$$T = \frac{2\pi\mu}{L} ab \tag{5.60}$$

In an ellipse, $b = a\sqrt{1-\epsilon^2}$. From Eq. (5.58), $1-\epsilon^2 = 1/aC$ and from Eq. (5.55) $C = \mu k / L^2$. Hence,

$$b^2 = a^2(1-\epsilon^2) = \frac{a^2}{aC} = \frac{aL^2}{\mu k} \tag{5.61}$$

With this value of b^2, Eq. (5.60) takes the form

$$T^2 = \frac{4\pi^2 \mu a^3}{k} \tag{5.62}$$

which is the statement of Kepler's third law.

We would be able to get the third law in an alternative form by replacing *m* by $m_1 m_2 /(m_1 + m_2)$ and k by its value $G m_1 m_2$:

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \tag{5.63}$$

As the mass of the planet m_1 compared to the mass of sun m_2 is very small,

$$T^2 \cong \frac{4\pi^2 a^3}{G m_2} \tag{5.64}$$

In this approximate expression the proportionality constant 4 p^2/Gm_2 is the same for all planets. Eq. (5.64) is fairly valid, except in the case of Jupiter which has a mass of about 0.1% of the mass of the sun.

The orbital eccentricities of the planets vary from 0.007 for Venus to 0.249 for Pluto. For Earth's orbit = 0.017, r_{min} = 145.6 10^6 km and r_{max} = 152 10^6 km. Comets generally have very high orbital eccentricities. Halley's comet has a value of = 0.967. The non-returning type comets have either parabolic or hyperbolic orbits.

5.8 LAW OF GRAVITATION FROM KEPLER'S LAWS

Kepler's laws paved the way for Newton to develop his law of gravitation. This can easily be proved. To start with, we will show that it is a central force and then proceed to prove that it is of the inverse square type. From Kepler's second law, Eqs. (5.21) and (5.22), we have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} = \text{constant} \tag{5.65}$$

$$\frac{d(r^2\dot{\theta})}{dt} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Since the acceleration perpendicular to the r-direction $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, the above can be written as

$$\frac{d(r^2\dot{\theta})}{dt} = ra_\theta \tag{5.66}$$

The time derivative of $r^2\dot{\theta}$ is zero since $r^2\dot{\theta}$ is a constant. Hence,

$$\frac{d(r^2\dot{\theta})}{dt} = ra_\theta = 0 \tag{5.67}$$

That is, the transverse acceleration on the planet is zero and therefore the force acting on the planet is a central one.

The force law can be determined from the differential equation of the orbit, Eq. (5.48), which can be written as

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu}{L^2}r^2F(r) \tag{5.68}$$

in which r is given by Kepler's first law, Eq. (5.56).

$$\frac{1}{r}=C(1+\in\cos\theta) \qquad C=\frac{\mu k}{L^2}$$

Replacing $\frac{1}{r}$ on the left hand side of Eq. (5.68), we get

$$C\frac{d^2}{d\theta^2}(1+\in\cos\theta)+C(1+\in\cos\theta)=-\frac{\mu}{L^2}r^2F(r) \tag{5.69}$$

$$C=-\frac{\mu}{L^2}r^2\ F(r)$$

$$F(r)=-\frac{CL^2}{\mu}\frac{1}{r^2}=-\frac{K'}{r^2} \tag{5.70}$$

where,

$$K'=\frac{CL^2}{\mu}=\text{constant} \tag{5.71}$$

The negative sign indicates that the force is one of attraction. From Eq. (5.60)

$$T^2=\frac{4\pi^2\mu^2}{L^2}\ a^2b^2 \tag{5.72}$$

From Eq. (5.61), $b^2 = a/C$. Substituting this value of b^2

$$T^2=4\pi^2\mu\left(\frac{\mu}{CL^2}\right)a^3=\frac{4\pi^2\mu a^3}{K'} \tag{5.72a}$$

Comparing this with Eq. (5.63) we get

$$\frac{\mu}{K'}=\frac{1}{G(m_1+m_2)} \quad \text{or} \quad \frac{m_1m_2}{(m_1+m_2)K'}=\frac{1}{G(m_1+m_2)}$$

$$K'=G\ m_1m_2 \tag{5.73}$$

With this value of K', Eq. (5.70) reduces to

$$F(r)=-\frac{G\ m_1m_2}{r^2}\ \hat{r} \tag{5.74}$$

which is the gravitational force of sun on a planet.

5.9 SATELLITE PARAMETERS

Today there are many satellites in orbit around the earth. The orbits of satellites are an interesting application of the central force problem. For circular orbits, the eccentricity $\in = 0$ and the satellite travels at a constant speed. In elliptical orbits, $0 < \in < 1$ and the speed of the satellite changes from position to position with maximum speed at the perigee and minimum at the apogee. The locations of some of the quantities in elliptical orbits are shown in Fig. 5.7.

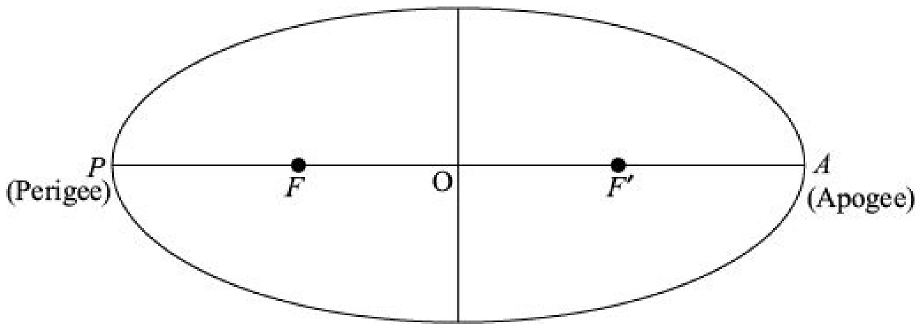


Fig. 5.7 Different parameters in elliptical orbits: Centre of the ellipse – O; Force centre – F ; Perigee distance – PF, Apogee distance – FA.

From the definition of eccentricity () we have

$$\in = \frac{\text{Interfocal distance}}{\text{Major axis}} = \frac{FF'}{PA} = \frac{FF'}{2a}$$

Hence,

$$\epsilon = \frac{\text{Interfocal distance}}{\text{Major axis}} = \frac{FF'}{PA} = \frac{FF'}{2a}$$

Hence,

$$OF = a\epsilon \tag{5.75a}$$

$$r_{\min} = PF = a - a\epsilon = a(1-\epsilon) \tag{5.75b}$$

$$r_{\max} = FA = a + a\epsilon = a(1+\epsilon) \tag{5.75c}$$

From Eqs. (5.55) and (5.57)

$$PF = r_{\min} = \frac{1}{C(1+\epsilon)} = \frac{L^2/\mu k}{1+\epsilon} \tag{5.76a}$$

$$FA = r_{\max} = \frac{1}{C(1-\epsilon)} = \frac{L^2/\mu k}{1-\epsilon} \tag{5.76b}$$

For completeness, some of the other parameters which we have already discussed are also listed here. From Eqs.(5.58), (5.61),(5.59) and (5.54) we have

The semi-major axis $$a = \frac{1}{C(1-\epsilon^2)} = \frac{L^2}{\mu k(1-\epsilon^2)} \tag{5.77a}$$

The semi-minor axis $$b = a\sqrt{(1-\epsilon^2)} \tag{5.77b}$$

Energy in an orbit $$E = -\frac{k}{2a} \tag{5.77c}$$

Angular momentum in an orbit $$L = \left[\frac{\mu k^2(\epsilon^2-1)}{2E}\right]^{1/2} \tag{5.77d}$$

Next, let us express the eccentricity, position and velocity of a satellite in terms of certain parameters at perigee. From Eq. (5.76a)

$$\epsilon = \frac{L^2}{\mu k\, r_{\min}} - 1 \qquad (5.78a)$$

If $\mathbf{v}_p$ is the velocity of the satellite at $r_{\min}$, $L = \mu \mathbf{v}_p\, r_{\min}$

$$\epsilon = \frac{\mu r_{\min} \mathbf{v}_p^2}{k} - 1 \qquad (5.78b)$$

Writing

$$\frac{k}{\mu r_{\min}} = \mathbf{v}_0^2 \qquad (5.78c)$$

$$\epsilon = \frac{\mathbf{v}_p^2}{\mathbf{v}_0^2} - 1 \qquad (5.78d)$$

For circular orbits as = 0 we see that $\mathbf{v}_0 = \mathbf{v}_p$. For $\mathbf{v}_p \geq \mathbf{v}_0$, the eccentricity is given by Eq. (5.78d). From Eq. (5.78a)

$$\frac{L^2}{\mu k} = r_{\min}(1+\epsilon) = r_{\min}\left(\frac{\mathbf{v}_p}{\mathbf{v}_0}\right)^2$$

Now the equation of the orbit, Eq. (5.56) can be written as

$$r = \frac{(\mathbf{v}_p / \mathbf{v}_0)^2\, r_{\min}}{1 + [(\mathbf{v}_p / \mathbf{v}_0)^2 - 1]\cos\theta} \qquad (5.79)$$

Since $$\frac{GMm}{R^2} = mg$$

$$k = GMm = mgR^2$$

where R is the radius of the earth and g is the acceleration due to gravity. The velocity of the satellite can be calculated from the relations

$$E = -\frac{k}{2a} \qquad \text{and} \qquad E = \frac{1}{2}m\mathbf{v}^2 + \left(\frac{-k}{r}\right)$$

$$\frac{1}{2}m\mathbf{v}^2 = E + \frac{k}{r} = \frac{k}{r} - \frac{k}{2a} = mgR^2\left(\frac{1}{r} - \frac{1}{2a}\right)$$

$$\mathbf{v} = \sqrt{2gR^2\left(\frac{1}{r} - \frac{1}{2a}\right)} \tag{5.80}$$

which is the velocity of the satellite in terms of *r*.

5.10 COMMUNICATION SATELLITES

Communication satellites are used for transmitting information from one part of the earth's surface to another. They are of two types, the *passive system* and the *active system*. A passive system simply reflects signals from the transmitting station to the receiving station. In the active system, the signal from the transmitter is received by the satellite and undergoes amplification in the satellite, and then it is again transmitted to the ground receiving station. In both the cases the satellite can be either stationary (synchronous satellite) or in motion with respect to the earth. Synchronous satellites are put into a circular orbit in the plane of the equator and the orbital period is selected to be one day, which is also the time the earth takes to turn once about its axis. Hence, these satellites move around their orbits in synchronous form with the rotation of the earth. For earth-based observers the satellite will be in a fixed position in the sky.

It is not difficult to find the height above the earth's surface at which all synchronous satellites must be placed in orbit. Since = 0, from Eq. (5.78c)

$$\mathbf{v}_0 = \sqrt{\frac{k}{\mu r_{\min}}}$$

Period of the satellite $$T = \frac{2\pi(R+H)}{\mathbf{v}_0}$$

where *R* is the radius of the earth and *H* is the altitude of the satellite. Substituting the value of $\mathbf{v}_0$ and remembering that $r_{\min} = R + H$

$$T = 2\pi(R+H)^{3/2}\sqrt{\frac{\mu}{k}}$$

$$R + H = \left(\frac{T}{2\pi}\sqrt{\frac{k}{\mu}}\right)^{2/3}$$

Since $M >> m, \mu \cong m.$ Replacing k by GMm

$$H = (GM)^{1/3}\left(\frac{T}{2\pi}\right)^{2/3} - R \qquad (5.81)$$

$$1 \text{ day} = 8.64 \times 10^4 \text{ s} \qquad G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$$

$$M = 5.98 \times 10^{24} \text{ kg} \qquad R = 6.38 \times 10^6 \text{ m}$$

Substituting these values in Eq. (5.81), we get H = 3.59 10^4 km which is a constant.

5.11 ORBITAL TRANSFERS

In this section we briefly investigate two types of orbital transfers: (i) transfer of a satellite in a circular orbit around the earth to an elliptical orbit around the earth, and (ii) sending space probes from one planet to another.

A satellite in a circular orbit of radius r_C around the earth can be sent into an elliptical orbit with a perigee distance r_C by a sudden blast of rockets at the proposed perigee. A rocket blast at perigee increases the velocity perpendicular to the radius vector only. The increase in velocity increases the energy E and the angular momentum. Consequently, the eccentricity increases from zero to positive value and the orbit changes from circular to elliptical (see Fig.5.8). This technique was followed in the Apollo moon mission.

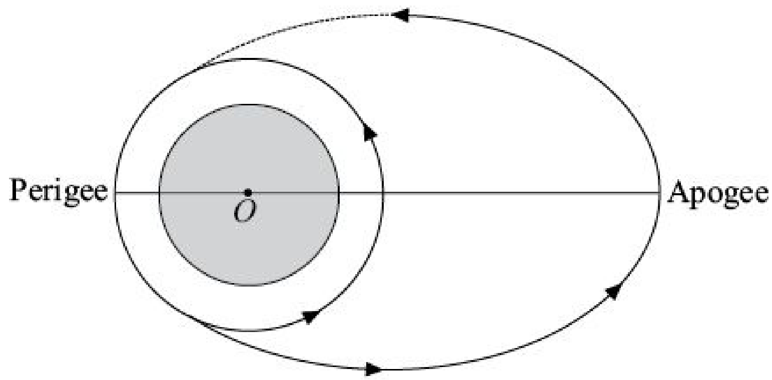


Fig. 5.8 Orbital transfer from circular to elliptical orbit around the earth.

The most efficient way of achieving the second type is to put the probe in an orbit, elliptical or circular, that joins the orbit of earth and that of the other planet. Such an orbit is called a **transfer orbit**. The situation is illustrated in Fig. 5.9. In figure the transfer orbit is dashed.

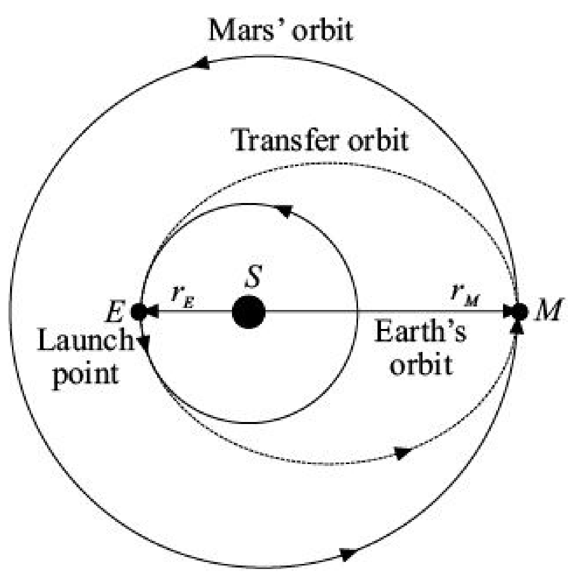


Fig. 5.9 The transfer orbit from Earth to Mars.

For discussion let us consider a space probe from earth to another planet, say Mars. For simplicity, let us assume that their orbits about the sun are circles of radii r_E and r_M. The transfer orbit is tangential to the earth's orbit at E and also tangential to Mars' orbit at M. The length of the major orbit of the transfer orbit $= r_E + r_M$. Let L and E be the angular momentum and energy of the earth's orbit and L and E be that of Mars' orbit, respectively with $E > E$ and $L > L$. To transfer a probe from Earth's to Mars' orbit, the probe should be given an acceleration at E to change the L and E values of earth, and once again an acceleration is given at M changing the values to L and E so that it can orbit around Mars.

For the earth to go around the sun (mass M_S) in an orbit of radius r_E and velocity $\mathbf{v}_0$.

$$\frac{m\mathbf{v}_0^2}{r_E} = \frac{GM_s m}{r_E^2} \quad \text{or} \quad \mathbf{v}_0 = \sqrt{GM_s / r_E} \tag{5.82}$$

The velocity $\mathbf{v}_0$ is also given by

$$\mathbf{v}_0 = \frac{2\pi r_E}{T_E} \tag{5.82a}$$

where T_E is the period of the orbital motion of Earth. At E, the probe is given a velocity $\mathbf{v}_1$ to be in the transfer orbit. In the transfer orbit, energy

$$E = \frac{-k}{\text{major axis}} = -\frac{GM_s m}{r_E + r_M}$$

Energy at E is also given by

$$E = \frac{1}{2} m\mathbf{v}_1^2 - \frac{GM_s m}{r_E}$$

Equating the two expressions

$$-\frac{GM_s m}{r_E + r_M} = \frac{1}{2} m\mathbf{v}_1^2 - \frac{GM_s m}{r_E}$$

$$\mathbf{v}_1^2 = \frac{2GM_s}{r_E} \frac{r_M}{(r_E + r_M)} = \mathbf{v}_0^2 \frac{2r_M}{r_E + r_M} \tag{5.83}$$

where Eq. (5.82) is used. The probe is given speed $\mathbf{v}_1$ at E so that it travels in an elliptical orbit, the transfer orbit whose apogee is M. Next let us evaluate the speed of the probe $\mathbf{v}_2$ when it reaches the apogee. By the law of conservation of angular momentum for the transfer orbit

$$\mathbf{v}_1 \times r_E = \mathbf{v}_2 \times r_M$$

$$\mathbf{v}_2 = \mathbf{v}_1 \frac{r_E}{r_M} \tag{5.84}$$

Taking T as the time period in the transfer orbit and using the result that T^2 is proportional to the cube of the major axis

$$\frac{T_E^2}{(2r_E)^3} = \frac{T^2}{(r_E + r_M)^3}$$

$$T = \left(\frac{r_E + r_M}{2r_E}\right)^{3/2} T_E \qquad (5.85)$$

Knowing r_E , r_M and T_E, we can calculate $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and T from Eqs. (5.82a), (5.83), (5.84) and (5.85).

Generally, $\mathbf{v}_2$ is less than the orbital speed of Mars, $\mathbf{v}_\mathrm{M}$. Hence, when the probe reaches the Martian orbit, the approaching Mars will overtake the probe. To avoid this when the probe arrives at *M*, the speed of the probe is increased from $\mathbf{v}_2$ to $\mathbf{v}_M$. If the probe transfer is to one of the inner planets, say Venus or Mercury, instead of increasing the speed, it has to be decreased from $\mathbf{v}_0$ to $\mathbf{v}_1$ to put the probe in a smaller transfer orbit. Again the probe has to be slowed down to the orbital speed of Venus.

5.12 SCATTERING IN A CENTRAL FORCE FIELD

Scattering is an important phenomenon in physics, since it is used to investigate different aspects in different areas in physics. The scattering of high energy *a* particles by positively charged atomic nuclei is a typical example of the motion of a particle in a central inverse square repulsive field. Such an experiment was first carried out by Geiger and Marsden and analyzed by Rutherford.

Consider a uniform beam of particles of same mass and energy and incident upon a centre of force. It will be assumed that the force falls off to zero for very large distances. The number of particles crossing a unit area placed normal to the beam in unit time is the intensity *I*, also called *flux density*. When a particle approaches a centre of force, it will either be attracted or repelled and its orbit will deviate from the incident straight line path. After passing the centre of force, the force acting on the particle will diminish, so that the orbit once again approaches a straight line. In general, the final direction of motion is not the same as the incident direction and the particle is said to be scattered. Fig 5.10

illustrates the scattering of an incident beam of particles by a scattering centre at O which is taken as the origin. Angle *f* is the angle between the incident and scattered directions and is called the *scattering angle*. The cross-section for scattering in a given direction, $\sigma(\Omega)$, is defined by

$$\sigma(\Omega)\, d\Omega = \frac{\text{number of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity } I} \quad (5.86)$$

where *d*W is an element of solid angle in the direction W. Often $\sigma(\Omega)$ is referred to as the **differential scattering cross-section** which has the dimension of area. Hence, the name scattering cross section. The **total cross-section** s_T is defined as the integral of $\sigma(\Omega)\, d\Omega$ over the entire solid angle 4*p*.

$$\sigma_T = \int_{4\pi} \sigma(\Omega)\, d\Omega \qquad (5.86a)$$

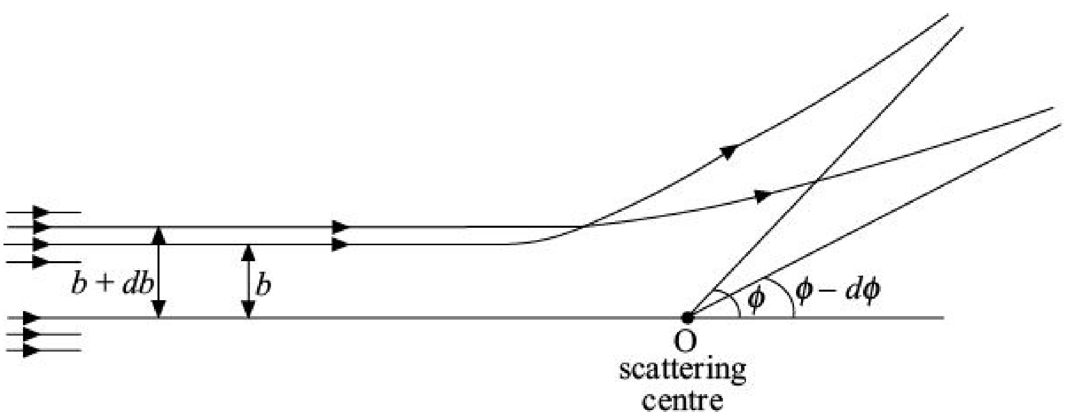


Fig. 5.10 Scattering of particles with impact parameters between *b* and *b* + *db* are scattered through angles between *f* and *f* – *df*.

The **impact parameter** *b* is defined as the perpendicular distance between the centre of force and the incident velocity direction. Fig. 5.10 shows particles with impact parameter between *b* and *b* + *db* being scattered through angles between ϕ and $\phi - d\phi$.

The number of particles incident in unit time with impact parameter lying between b and $b + db = I \times 2\pi b\, db$ (5.87)

The solid angle $d\Omega$ lying between ϕ and $\phi - d\phi$ is given by

$$d\Omega = \frac{2\pi r \sin\phi \times r d\phi}{r^2} = 2\pi \sin\phi d\phi \quad (5.88)$$

where r is the radius vector of the particle. From the definition of $\sigma(\Omega)$ we have

The number of particles scattered into solid angle $d\Omega$ in unit time

$$= \sigma(\phi)\, d\Omega I$$
$$= \sigma(\phi)\, 2\pi \sin\phi\, d\phi \times I \quad (5.89)$$

The number of particles scattered in unit time into the solid angle between ϕ and $\phi - d\phi$ must be equal to the number of incident particles in unit time with impact parameter between b and $b + db$. Hence,

$$\sigma(\phi)\, 2\pi \sin\phi\, d\phi \times I = I\, 2\pi b\, db$$

$$\sigma(\phi) = -\frac{b\, db}{\sin\phi\, d\phi} \quad (5.90)$$

The negative sign is introduced because an increase in impact parameter means less force is exerted on the particle, resulting in a decrease *df* in the scattering angle. Equation (5.90) is a general result valid for both repulsive and attractive inverse square fields.

To illustrate the procedure, let us consider the scattering of charged particles by a Coulomb field. Let the scatterer have a charge *Ze* and the incident particles a charge $Z'e$. The force between the charges is repulsive and is given by

$$f(r) = \frac{ZZ'e^2}{r^2} \quad (5.91)$$

For convenience the factor $4\pi \in_0$ is left out from the denominator. The corresponding potential $V = \frac{ZZ'e^2}{r}$ (5.92)

In Section 5.6, while solving the differential equation to get the equation of the orbit, we used $V(r) = -k/r$. However, here V is positive. Hence, the results of Section 5.6 can be taken over here with the change

$$k = -ZZ'e^2 \tag{5.93}$$

With this value of k, Eq. (5.52) reduces to

$$\frac{1}{r} = -\frac{\mu ZZ'e^2}{L^2}[1 + \epsilon \cos(\theta - \theta_0)] \tag{5.94}$$

$$\epsilon = \left(1 + \frac{2EL^2}{\mu Z^2 Z'^2 e^4}\right)^{1/2} \tag{5.95}$$

which is the eccentricity of the conic. As the constant θ_0 simply determines the orientation of the orbit, we can select $\theta_0 = \pi$ which will make the orbit symmetric about the direction of the periapsis. This reduces Eq. (5.94) to

$$\frac{1}{r} = \frac{\mu ZZ'e^2}{L^2}(\epsilon \cos\theta - 1) \tag{5.96}$$

Next let us see more about the eccentricity. If v_0 is the incident speed of the particle, its angular momentum is given by

$$L = \mu v_0 b \tag{5.97}$$

When the incident particle is far away from the scattering centre, the influence of the scatterer is not felt and therefore the total energy E of the particle is the same as the kinetic energy: $E = \frac{1}{2}\mu v_0^2$ or $v_0^2 = \frac{2E}{\mu}$ (5.98)

With this value of v_0^2, Eq. (5.97) takes the form

$$L^2 = 2\mu E b^2$$

Substituting this value of L^2 in Eq. (5.95), we have

$$\epsilon = \left[1 + \left(\frac{2Eb}{ZZ'e^2}\right)^2\right]^{1/2} \tag{5.99}$$

It is evident from Eq. (5.99) that the eccentricity $\epsilon > 1$. Hence, the path of the particle will be a hyperbola. Fig. 5.10 shows the orbit parameters and the scattering angle *f*. From Fig. 5.11, we have

$$\phi = \pi - 2\theta \quad \text{or} \quad \theta = \frac{\pi}{2} - \frac{\phi}{2}$$

$$\cos\theta = \sin\frac{\phi}{2} \tag{5.100}$$

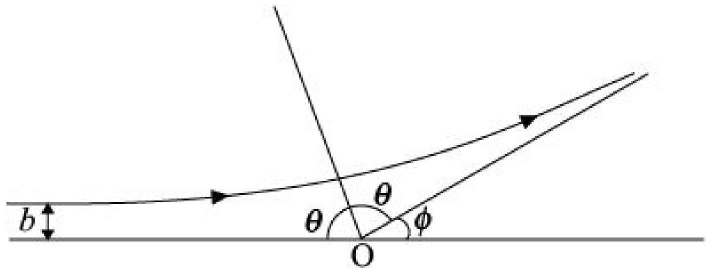


Fig. 5.11 Angle *q* and scattering angle *f* in repulsive scattering.

When *r* is very large, from Eq. (5.96) $\epsilon \cos\theta - 1 = 0 \quad \text{or} \quad \cos\theta = 1/\epsilon$. Hence,

$$\cos\theta = \sin\frac{\phi}{2} = \frac{1}{\epsilon} \quad \text{or} \quad \operatorname{cosec}\frac{\phi}{2} = \epsilon$$

$$\cot^2\frac{\phi}{2} = \operatorname{cosec}^2\frac{\phi}{2} - 1 = \epsilon^2 - 1 \tag{5.101}$$

Substituting the value of $\epsilon^2 - 1$ from Eq. (5.99), we get

$$b = \frac{ZZ'e^2}{2E}\cot\frac{\phi}{2}$$

$$db = \frac{1}{2}\left(\frac{ZZ'e^2}{2E}\right)\operatorname{cosec}^2\frac{\phi}{2}\,d\phi$$

Substituting these values in Eq. (5.90), we have

$$\sigma(\phi) = \frac{\left(ZZ'e^2/2E\right)^2 \cot\frac{\phi}{2}\operatorname{cosec}^2\frac{\phi}{2}\,d\phi}{4\sin\frac{\phi}{2}\cos\frac{\phi}{2}\,d\phi}$$

$$= \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \frac{1}{\sin^4\frac{\phi}{2}} \tag{5.102}$$

which is Rutherford's scattering formula for *a*-particle scattering. Non-relativistic quantum mechanics also gives the same result.

The assumption that no incident particle interacts with more than one target nucleus is valid if the scattering angle is not too small. An interesting feature of the expression is the appearance of the factor $ZZ'e^2$ as square. This indicates that the distribution of the scattered particles is the same for an attractive force as for a repulsive force.

5.13 SCATTERING PROBLEM IN LABORATORY CO-ORDINATES

The laboratory co-ordinate system is the one where the incident particle moves in and the scatterer is at rest. The actual measurements are made in this system. The scattering angle measured in the laboratory, denoted by *c*, is the angle between the final and initial directions of the scattered particle. However, in general the scatterer is not fixed but recoils from its position as a result of scattering. The scattering angle *f*, calculated from the equivalent one-body problem, is the angle between the final and initial directions of the relative vector between the two particles. The two angles *c* and *f* would be the same only if the scatterer is at rest throughout. In general, the two are different as shown in Fig. 5.12(a).

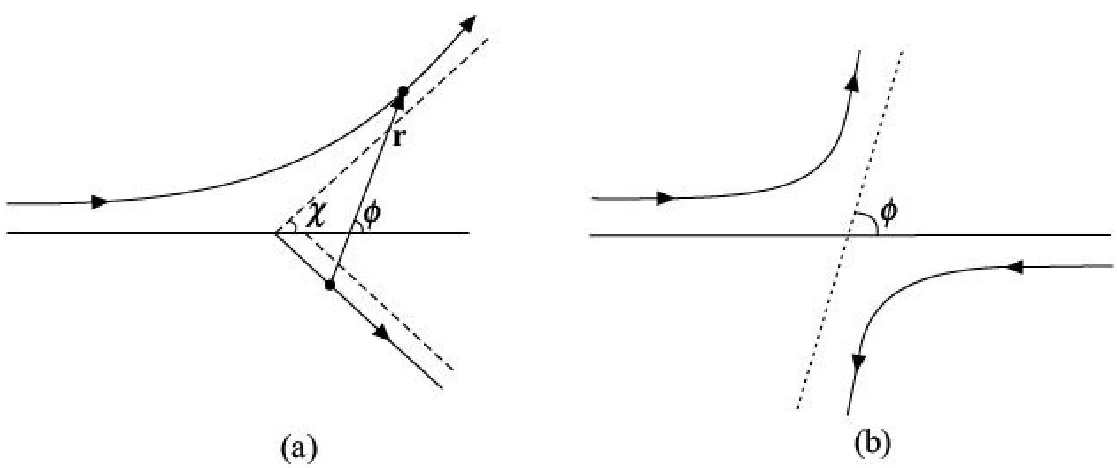


Fig. 5.12 Scattering of two particles: (a) as viewed in the laboratory system; (b) as viewed in the centre of mass system.

In the centre of mass system, the centre of mass is always at rest and is taken as the origin. In this system the total linear momentum is zero and therefore the two particles always move with equal and opposite momenta, as shown in Fig.5.12 (b). Before scattering the particles are moving directly toward each other; afterwards they are moving directly away from each other.

Next, we shall derive the relation connecting the two scattering angles *f* and *c*. Let $\mathbf{r}_1$ and $\mathbf{v}_1$ be respectively the position and velocity after scattering of the incident particle in the laboratory system and $\mathbf{r}_1'$ and $\mathbf{v}_1'$ the respective position and velocity after scattering of the particle in the centre of mass system. Let **R** and **V** be the respective position and velocity of the centre of mass in the laboratory system.

At any instant by definition (see Fig.5.1)

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_1'$$

Hence,

$$\mathbf{v}_1 = \mathbf{V} + \mathbf{v}_1' \qquad (5.103)$$

This vector relation is illustrated in Fig. 5.13 at a time after scattering. At that time the vector $\mathbf{v}_1$ makes an angle χ and $\mathbf{v}_1'$ makes an angle ϕ with the vector **V** which is along the initial direction. From Fig. 5.13

$$\tan\chi = \frac{AB}{OB} = \frac{\mathbf{v}_1'\sin\phi}{\mathbf{v}_1'\cos\phi + \mathbf{V}} = \frac{\sin\phi}{\cos\phi + (\mathbf{V}/\mathbf{v}_1')} \qquad (5.104)$$

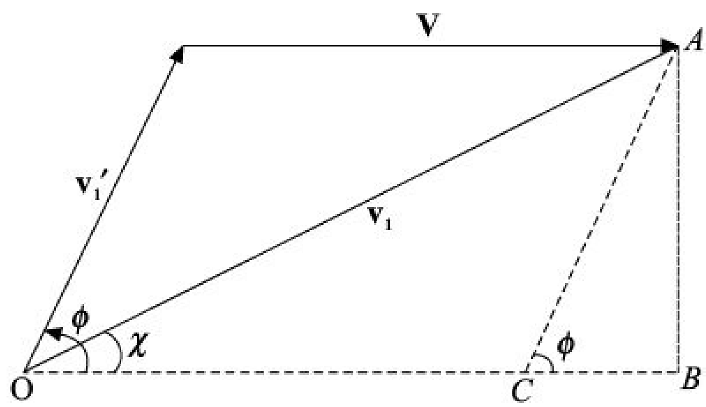


Fig. 5.13 Relation between the velocities in the centre of mass and laboratory co-ordinates.

From Eq. (5.5) we have

$$\mathbf{r}_1' = \frac{m_2 \mathbf{r}}{m_1 + m_2}$$

$$\mathbf{v}_1' = \frac{m_2 \mathbf{v}}{m_1 + m_2} = \frac{\mu \mathbf{v}}{m_1} \quad (5.105)$$

where $\mathbf{v}$ is the relative velocity after the collision. Since the scatterer is initially at rest, by the law of conservation of linear momentum

$$(m_1 + m_2)\mathbf{V} = m_1 \mathbf{v}_0$$

$$\mathbf{V} = \frac{m_1 \mathbf{v}_0}{(m_1 + m_2)} = \frac{\mu \mathbf{v}_0}{m_2} \quad (5.106)$$

Substituting these values of $\boldsymbol{V}$ and $\mathbf{v}_1'$ in Eq. (5.104), we get

$$\tan\chi = \frac{\sin\phi}{\cos\phi + \rho} \quad (5.107)$$

where,

$$\rho = \frac{\mu \mathbf{v}_0}{m_2 \mathbf{v}_1'} = \frac{m_1 \mathbf{v}_0}{m_2 \mathbf{v}} \quad (5.108)$$

When the collision is elastic the conservation of the total kinetic energy of the two particles leads to $\mathbf{v} = \mathbf{v}_0$. In that case

$$\rho = \frac{m_1}{m_2} \text{ (elastic scattering)} \quad (5.109)$$

If $m_2 >> m_1$, $\rho \to 0$ and we get

$$\tan\chi \cong \tan\phi \quad \text{or} \quad \chi = \phi \quad (5.110)$$

which is the case for a fixed scattering centre. If the scattering is inelastic, $\mathbf{v} \neq \mathbf{v}_0$ and we need detailed calculations based on the amount of energy transfer for the evaluation of $\mathbf{v}/\mathbf{v}_0$.

For the interpretation of the results the relation between *s* (*f*) and *s*(*c*) is required; here *s*(*c*) is the differential scattering cross-section expressed in terms of the scattering angle in the laboratory system. This can be obtained by using the condition that the number of particles scattered into a given element of a solid angle must be the same, both in the centre of mass and the laboratory co-ordinate systems. That is,

$$2\pi I\ \sigma(\phi)\sin\phi\, d\phi = 2\pi I\, \sigma'(\chi)\sin\chi\, d\chi$$

$$\sigma'(\chi) = \sigma(\phi)\frac{\sin\phi\, d\phi}{\sin\chi\, d\chi} = \sigma(\phi)\frac{d(\cos\phi)}{d(\cos\chi)} \tag{5.111}$$

From Fig. 5.13, we have

$$\mathbf{v}_1^2 = \mathbf{v}_1'^2 + \mathbf{V}^2 + 2\mathbf{v}_1'\mathbf{V}\cos\phi \tag{5.112}$$

and

$$\mathbf{v}_1\cos\chi = \mathbf{V} + \mathbf{v}_1'\cos\phi \tag{5.113}$$

Substituting for $\mathbf{v}_1$ from Eq. (5.112) , Eq. (5.113) becomes

$$\cos\chi = \frac{\mathbf{V} + \mathbf{v}_1'\cos\phi}{(\mathbf{v}_1'^2 + \mathbf{V}^2 + 2\mathbf{v}_1'\mathbf{V}\cos\phi)} \tag{5.114}$$

Replacing V using Eq. (5.106) and introducing ρ which is defined by Eq. (5.108), we get

$$\cos\chi = \frac{\rho + \cos\phi}{(1 + 2\rho\cos\phi + \rho^2)^{1/2}} \tag{5.115}$$

Differentiating

$$\frac{d(\cos\chi)}{d(\cos\phi)} = \frac{1}{(1 + 2\rho\cos\phi + \rho^2)^{1/2}} - \frac{(\rho + \cos\phi)\rho}{(1 + 2\rho\cos\phi + \rho^2)^{3/2}}$$

$$= \frac{1 + \rho\cos\phi}{(1 + 2\rho\cos\phi + \rho^2)^{3/2}} \tag{5.116}$$

Combining Eqs. (5.111) and (5.116) we have

$$\sigma'(\chi) = \sigma(\phi)\frac{(1 + 2\rho\cos\phi + \rho^2)^{3/2}}{1 + \rho\cos\phi} \tag{5.117}$$

There are two interesting cases when the scattering is elastic, or $\rho = m_1/m_2$.

$\boldsymbol{m_2 > m_1}$**:** In the case of Rutherford α scattering, $m_1 = 4$ atomic units and m_2 is usually 100 atomic units or larger. Under this condition, ρ is very small and can be taken as zero. In that case Eq. (5.117) reduces to

$$\sigma'(\chi) = \sigma(\phi) \tag{5.118}$$

in conformity with the result in Eq. (5.110).

$\boldsymbol{m_1 = m_2, \rho = 1}$**:** A typical example of this case is neutron-proton scattering. From Eq. (5.107)

$$\tan\chi = \frac{\sin\phi}{\cos\phi + 1} = \frac{2\sin{}^{\phi}\!/_{2}\ \cos{}^{\phi}\!/_{2}}{2\cos^2{}^{\phi}\!/_{2}} = \tan{}^{\phi}\!/_{2}$$

$$\chi = {}^{\phi}\!/_{2} \tag{5.119}$$

Substitution of Eq. (5.119) in Eq. (5.117) leads to

$$\sigma'(\chi) = 4\sigma(2\chi)\cos\chi \qquad \chi \le \frac{\pi}{2}, \rho = 1 \tag{5.120}$$

Thus, with equal masses, scattering angles greater than 90° is not possible in the laboratory system. The entire scattering takes place in the forward hemisphere.

WORKED EXAMPLES

Example 5.1 The orbit of a particle of mass *m* moving in a central force is given by $r = kq^{2}$, where *k* is a constant. Find the law of force.

Solution: In the central force problem, the equation of the orbit is given by Eq. (5.48)

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m}{L^2} r^2 F(r) \tag{i}$$

It is given that $r = k\theta^2$ or $\frac{1}{r} = \frac{1}{k\theta^2}$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{k\theta^2}\right) = -\frac{2}{k\theta^3}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{6}{k\theta^4} = \frac{6k}{r^2} \qquad \text{(ii)}$$

Substituting Eq. (ii) in Eq. (i), we get

$$\frac{6k}{r^2} + \frac{1}{r} = -\frac{m}{L^2}r^2 F(r)$$

$$F(r) = -\frac{L^2}{m}\left(\frac{6k}{r^4} + \frac{1}{r^3}\right) \qquad \text{(iii)}$$

which is the law of force.

Example 5.2 A particle moves in a circular orbit in a force field $F(r) = -k/r^2$. Suddenly k becomes $k/2$ without change in velocity of the particle. Show that the orbit becomes parabolic.

Solution: In elliptical orbits, from Eq. (5.59) we have the total energy $E = -k/2a$. In the case of = 0 an elliptical orbit reduces to a circle, the semi-major axis a equals semi-minor axis b and is just the radius of the circle r. For the circular orbit, the total energy E, potential energy V and kinetic energy T are given by $E = -\frac{k}{2r}$ $V = -\frac{k}{r}$ and $T = \frac{k}{2r}$

When k becomes $k/2$ there is no change in velocity. Hence, kinetic energy remains the same, but potential energy changes.

New potential energy $= -\frac{k}{2r}$

Total energy $E = \frac{k}{2r} + \left(-\frac{k}{2r}\right) = 0$

Then the eccentricity of the orbit is 1 which is a parabola.

Example 5.3 A particle moves in a circular orbit of diameter b in a central force field. If the centre of attraction is on the circumference itself, find the law of force.

Solution: In a central field, the differential equation of the orbit, Eq. (5.48), is given by

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{m}{L^2}r^2\,F(r) \qquad \text{(i)}$$

In Fig. 5.14, O is the centre of force, and *A* is the position of the particle. The co-ordinates of the particle are *r* and *q*. From the figure

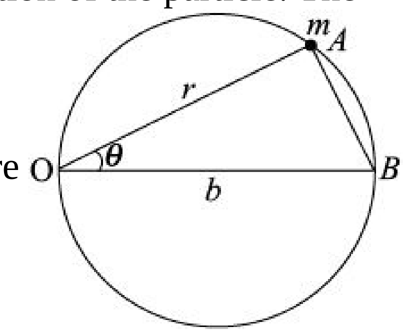


Fig. 5.14 The circular orbit of the particle.

$$r = b\cos\theta \qquad \text{(ii)}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right)=\frac{d}{d\theta}\left(\frac{\sec\theta}{b}\right)=\frac{1}{b}\sec\theta\tan\theta$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)=\frac{1}{b}(\sec\theta\tan^2\theta+\sec^3\theta) \qquad \text{(iii)}$$

Substituting Eq. (iii) in Eq. (i), we get

$$\frac{1}{b}(\sec\theta\tan^2\theta+\sec^3\theta)+\frac{\sec\theta}{b}=-\frac{m}{L^2}b^2\cos^2\theta\,F(r) \qquad \text{(iv)}$$

$$\frac{1}{b}[\sec\theta(\sec^2\theta-1)+\sec^3\theta]+\frac{\sec\theta}{b}=-\frac{m}{L^2}b^2\cos^2\theta\,F(r)$$

$$\frac{2\sec^3\theta}{b}=-\frac{m}{L^2}b^2\cos^2\theta\,F(r)$$

$$F(r)=\frac{-2L^2\sec^5\theta}{mb^3}=\frac{-2L^2b^2}{mr^5}=\frac{K}{r^5} \qquad \text{(v)}$$

where *K* is a constant.

Example 5.4 The eccentricity () of earth's orbit around sun is 1/60. Show that the time taken for travel of the arc ABC is about 2 days more than the time it takes to trace CDA (see Fig. 5.15).

Solution: While travelling the arc CDA the radius vector moves from C to D and then to A. During this, the area of the triangle (shaded) is left out. This area is included while travelling the arc ABC. Since areal velocity is constant, this additional area will certainly take some time. From Eqs. (5.57) and (5.58) we have

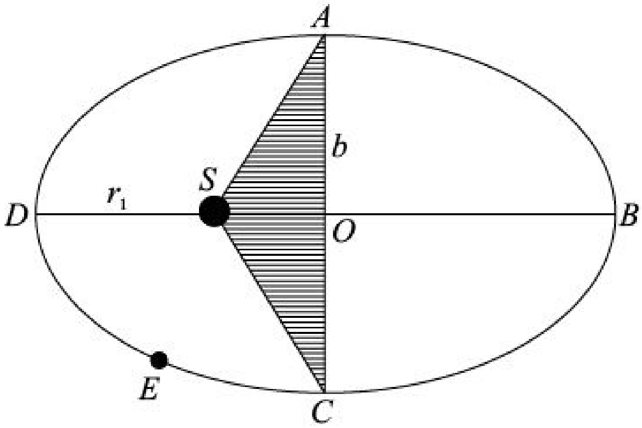


Fig. 5.15 Earth's orbit around the sun.

$$r_1 = \frac{1}{C(1+\epsilon)} \quad \text{and} \quad a = \frac{1}{C(1-\epsilon^2)}$$

$$SO = a - r_1 = \frac{1}{C(1-\epsilon^2)} - \frac{1}{C(1+\epsilon)}$$

$$= \frac{\epsilon}{C(1-\epsilon^2)} = a\epsilon$$

Additional area $= 2 \times \frac{1}{2} \times b \times a\epsilon = ba\epsilon$

Time required to sweep area $ab\epsilon = \frac{ab\epsilon}{\pi ab}$ years

$$= \frac{\epsilon}{\pi} \text{ years}$$

$$= \frac{1 \times 365 \text{ days}}{60 \times \pi} \cong 2 \text{ days}$$

Example 5.5 A satellite in an elliptical orbit around the earth has the equation

$$\frac{8200 \text{ km}}{r} = 1 + 0.2\cos\theta$$

Find: (i) the values of semi-major and semi-minor axes; (ii) the period of the satellite; (iii) the altitude of perigee and, apogee and (iv) the velocity of the satellite at perigee and apogee. Radius of the earth = 6380 km, mass of the earth = 5.97 10^{24}kg, gravitational constant G = 6.67 10^{-11}Nm^2/kg^2.

Solution: From Eq. (5.56), we have

$$\frac{1/C}{r} = 1 + \epsilon \cos\theta \qquad \text{(i)}$$

Equation (5.58) gives $$\frac{1}{C} = a(1-\epsilon^2) \qquad \text{(ii)}$$

Combining the two equations we have

$$\frac{a(1-\epsilon^2)}{r} = 1+\epsilon\cos\theta \qquad \text{(iii)}$$

The given equation is $$\frac{8200 \text{ km}}{r} = 1+0.2\cos\theta \qquad \text{(iv)}$$

Comparing Eqs. (iii) and (iv) we get

$$a\,(1-\epsilon^2) = 8200 \text{km} \quad \text{and} \quad \epsilon = 0.2 \qquad \text{(v)}$$

(i) The semi-major axis $a = \dfrac{8200 \text{km}}{0.96} = 8541.7 \text{km}$

The semi-minor axis $b = a\sqrt{1-\epsilon^2} = 8369.1 \text{km}$

(ii) Period of the satellite $T = \dfrac{2\pi a^{3/2}}{\sqrt{GM_e}} = 2$ hrs 11 m 2s

(iii) $$r_{\min} = a(1-\epsilon) = 8541.7 \text{ km} \times 0.8 = 6833.4 \text{ km.}$$

$$r_{\max} = a(1+\epsilon) = 8541.7 \text{ km} \times 1.2 = 10250.0 \text{ km.}$$

Altitude of the perigee = 6833.4 km – R_E = 453.4 km

Altitude of the apogee = 10250.0 km – R_E = 3870.0 km

(iv) Velocity at the perigee and apogee

$$\mathbf{v}_\theta = r\dot{\theta} = \frac{L}{mr} \quad \text{[from Eq. (5.18)]}$$

$$\frac{L}{m} = \frac{2A}{T} \quad \text{[from Eq. (5.23)]}$$

Hence,

$$\mathbf{v}_\theta = \frac{2A}{Tr} = \frac{2\pi ab}{Tr}$$

$$\text{Velocity at perigee } = \frac{2\pi ab}{Tr_{\min}} = 30111 \text{ km/h}$$

$$\text{Velocity at apogee } = \frac{2\pi ab}{Tr_{\max}} = 20074 \text{ km/h}$$

Example 5.6 A satellite of mass 2500 kg is going around the earth in an elliptic orbit. The altitude at the perigee is 1100 km, while at the apogee it is 3600 km. Calculate (i) the value of the semi major axis, (ii) the eccentricity of the orbit, (iii) the energy of the satellite, (iv) the angular momentum of the satellite, and (v) the satellite's speed at perigee ($\mathbf{v}_p$) and at apogee ($\mathbf{v}_a$). Radius of the earth = 6400 km, gravitational constant = $6.67\ \ 10^{-11}$ $N.m^2/kg^2$, mass of the earth = $5.97\ \ 10^{24}$ kg.

Solution: We have

(i) r_{min} = radius of the earth + altitude at perigee = 7500 km

r_{max} = radius of the earth + altitude at apogee = 10000 km

$$a = \frac{r_{min} + r_{max}}{2} = \frac{17500}{2}\,\text{km} = 8750\,\text{km}$$

(ii)
$$\in = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{r_{max} - r_{min}}{2a} = \frac{2500\,\text{km}}{17500\,\text{km}} = \frac{1}{7}$$

(iii)
$$E = -\frac{k}{2a} = -\frac{GMm}{2a} = -5.69 \times 10^{10}\,\text{J}$$

(iv)
$$\in = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

$$L^2 = -\frac{\mu k^2 (1 - \in^2)}{2E} = -\frac{m(GMm)^2(1 - \in^2)}{2E}$$

Substituting the values and simplifying

$$L = 1.46 \times 10^{14}\ \text{kg.m}^2/\text{s}$$

(v) Conservation of angular momentum gives

$$L = m\,\text{v}_p\, r_{min} = m\,\text{v}_a\, r_{max}$$

$$\text{v}_p = \frac{L}{m r_{min}} = \frac{1.46 \times 10^{14}\,\text{kg.m}^2/\text{s}}{2500\,\text{kg}\,(7500 \times 10^3\,\text{m})} = 7786.7\ \text{m/s}$$

$$\text{v}_a = \frac{L}{m\, r_{max}} = \frac{1.46 \times 10^{14}\ \text{kg.m}^2/\text{s}}{2500\,\text{kg}\,(10000 \times 10^3\, m)} = 5840.0\ \text{m/s}$$

Example 5.7 A spacecraft in a circular orbit of radius r_C around the earth was put in an elliptical orbit by firing a rocket. If the speed of the spacecraft increased by 12.5% by the sudden firing of the rocket, (i) What is the equation of the new orbit? (ii) What is its eccentricity? (iii) What is the apogee distance?

Solution: Let v_C be the speed in the circular orbit. The speed after firing of rocket $v_0 = v_c + 0.125\, v_c = 1.125\, v_c$

(i) From Eq. (5.79) the equation of orbit is given by

$$r = \frac{(1.125)^2\, r_c}{1 + [(1.125)^2 - 1]\cos\theta} = \frac{1.27\, r_c}{1 + 0.27\cos\theta}$$

(ii) Eccentricity $\epsilon = \left(\frac{v_p}{v_0}\right)^2 - 1 = (1.125)^2 - 1 = 0.27$

(iii) At the apogee, $\theta = \pi$ and r is r_{max},

$$r_{max} = \frac{1.27\, r_c}{1 - 0.27} = 1.74\, r_c$$

Example 5.8 A spacecraft launched from earth has to be put into an orbit around Mars. Calculate (i) the speed of the spacecraft around the earth, (ii) the speed of the spacecraft at the launch point in the transfer orbit, (iii) the speed of the craft when it arrives at Mars, and (iv) the time taken by the craft for the Earth – Mars trip. Radius of the Earth orbit = $1.49 \;\; 10^8$ km, radius of the Mars orbit = $2.265 \;\; 10^8$ km.

Solution:

(i) Time period of the orbital motion of the earth $T_E = 3.15 \times 10^7$s

Speed around the earth $$v_0 = \frac{2\pi(1.49 \times 10^8 \text{ km})}{3.15 \times 10^7 \text{ s}} = 29.7 \text{ km/s}$$

(ii) The speed of the spacecraft at the perigee of the transfer orbit is given by Equation (5.83)

$$v_1^2 = \frac{2r_M}{r_E + r_M} v_0^2 = \frac{(4.53 \times 10^8 \text{ km})(29.7 \text{ km/s})^2}{3.755 \times 10^8 \text{ km}} = 1064.15 \text{ km}^2/\text{s}^2$$

$$v_1 = 32.62 \text{ km/s}$$

which is the launch velocity in the co-ordinate frame of the sun.

(iii) From Eq. (5.84)

$$v_2 = \frac{r_E}{r_M} v_1 = \frac{(1.49 \times 10^8 \text{km})(32.62 \text{ km/s})}{2.265 \times 10^8 \text{ km}} = 21.46 \text{ km/s}$$

(iv) From Eq.(5.85), the period in the transfer orbit is

$$T = \left(\frac{r_E + r_M}{2r_E}\right)^{3/2} T_E = \left(\frac{3.755 \times 10^8 \text{ km}}{2.98 \times 10^8 \text{ km}}\right)^{1/2} (3.15 \times 10^7 \text{s}) = 4.46 \times 10^7 \text{s}$$

Time for Earth-Mars trip $= \dfrac{T}{2} = 2.23 \times 10^7 \text{s} = 258.1$ days

Note: We can calculate the orbital speed of Mars and then by how much the speed has to be decreased when it reaches Mars.

Period of Mars around the sun $= 1.88$ yrs $= 5.93 \times 10^7$s

Orbital speed of Mars $$v_M = \frac{2\pi r_M}{5.93 \times 10^7 \text{s}} = \frac{2\pi(2.265 \times 10^8 \text{ km})}{5.93 \times 10^7 \text{s}} = 24 \text{ km/s}$$

Thus, when arriving at Mars, the craft's speed has to be increased by (24.0 – 21.46) km/s = 2.54 km/s in order for the spacecraft to go into orbit around Mars.

Example 5.9 Consider scattering of particles by a rigid sphere of radius *R* and calculate the differential and total cross-sections.

Solution: Since the sphere is rigid, the potential outside the sphere is zero and that inside is . Fig. 5.17 illustrates the scattering by a rigid sphere. A particle with impact parameter $b > R$ will not be scattered. If $b < R$, due to the law of conservation of momentum and energy a particle incident at an angle *a* with the normal to the surface of the sphere will be scattered off on the other side of the normal at the same angle *a* (see Fig. 5.17).

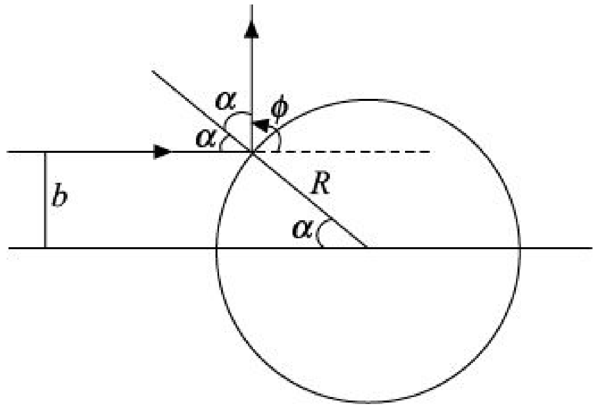


Fig. 5.16 Scattering by a rigid sphere.

From figure

$$\sin\alpha = \frac{b}{R} \qquad \text{and} \qquad \phi = \pi - 2\alpha$$

$$\alpha = \frac{\pi - \phi}{2} \qquad \text{or} \qquad \sin\alpha = \sin\frac{\pi - \phi}{2} = \cos\frac{\phi}{2}$$

Equating the two expressions for $\sin\alpha$

$$b = R\cos\frac{\phi}{2}$$

Substituting this value of b in Eq. (5.90)

$$\sigma(\phi) = -\frac{b}{\sin\phi}\frac{db}{d\phi} = \frac{R^2}{4}$$

which is independent of *f* and incident energy.

$$\sigma_T = \int_{4\pi} \sigma(\Omega)d\Omega = 2\pi \int_0^{\pi} \sigma(\phi)\sin\phi\, d\phi$$

$$= 2\pi\frac{R^2}{4}[-\cos\phi]_0^{\pi} = \pi R^2$$

Example 5.10 Derive an expression for the velocity of an earth satellite in terms of its radius vector r, the semi major axis a of its elliptical orbit, the mass of the earth M and the gravitational constant G. Hence, obtain its velocity at perigee and apogee if a = 27411.8 km and eccentricity $\in = 0.758.$ The value of

$GM = 39.82 \times 10^{13}$ m^3/s^2.

Solution: Total energy E of the satellite is given by

$$E = \frac{1}{2} m\mathrm{v}^2 + \left(\frac{-k}{r}\right) \qquad k = GMm$$

where m is the mass of the satellite and v its velocity. Energy E is also given by

$$E = \frac{-k}{2a}$$

Equating the two expressions for energy and substituting for k

$$-\frac{GMm}{2a} = \frac{1}{2} m\mathrm{v}^2 - \frac{GMm}{r}$$

$$\mathrm{v}^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

At perigee, $r = a(1 - \in)$. Hence, velocity at perigee ($\mathbf{v}_p$) is,

$$\mathrm{v}_p^2 = GM\left(\frac{2}{a(1-\in)} - \frac{1}{a}\right) = \frac{GM}{a}\left(\frac{1+\in}{1-\in}\right)$$

At apogee, $r = a(1-\in)$ and velocity at apogee $\mathbf{v}_a$ is given by

$$\mathrm{v}_a^2 = \frac{GM}{a}\left(\frac{1-\in}{1+\in}\right)$$

Substituting the values of GM, a and $\in$ and simplifying

$$\mathrm{v}_p = 10.15 \text{ km/s} \qquad \text{and} \qquad \mathrm{v}_a = 1.43 \text{ km/s}$$

Example 5.11 Find the law of force if a particle under central force moves along the curve $r = a(1 + \cos\theta)$.

Solution: The differential equation of the orbit is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{mr^2}{L^2}F(r) \qquad \text{(i)}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right)=\frac{d}{d\theta}\left(\frac{1}{a(1+\cos\theta)}\right)=\frac{\sin\theta}{a(1+\cos\theta)^2} \qquad \text{(ii)}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)=\frac{d}{d\theta}\left\{\frac{\sin\theta}{a(1+\cos\theta)^2}\right\}=\frac{\cos\theta}{a(1+\cos\theta)^2}+\frac{2\sin^2\theta}{a(1+\cos\theta)^3}$$

$$=\frac{a\cos\theta}{a^2(1+\cos\theta)^2}+\frac{2a^2(1-\cos^2\theta)}{a^3(1+\cos\theta)^3}=\frac{r-a}{r^2}+\frac{2a^2-2(r-a)^2}{r^3}$$

$$=\frac{r-a}{r^2}+\frac{-2r^2+4ar}{r^3}=\frac{1}{r}-\frac{a}{r^2}-\frac{2}{r}+\frac{4a}{r^2}$$

$$=-\frac{1}{r}+\frac{3a}{r^2} \qquad \text{(iii)}$$

Substituting Eq. (iii) in Eq.(i)

$$-\frac{1}{r}+\frac{3a}{r^2}+\frac{1}{r}=-\frac{mr^2}{L^2}F(r)$$

$$F(r)=-\frac{3aL^2}{mr^4}$$

which is the law of force.

REVIEW QUESTIONS

1. What is a central force? Are all central forces conservative?
2. In central force motion, the conservation of angular momentum implies the constancy of the areal velocity. Prove.
3. Outline the general properties of central force motion.
4. What is the first integral of the central force motion? Explain with an example.
5. In central force motion, obtain the energy equation in the form

$$E=\frac{1}{2}\mu\dot{r}^2+\frac{\mathbf{L}^2}{2\mu r^2}+V(r) \qquad \mathbf{L} = \text{angular momentum}$$

6. In the case of inverse square law force field, if the orbit is circular prove that the potential energy is twice the total energy.
7. How does the value of eccentricity and energy determine the shape of the orbit in a central force problem?
8. Consider a particle of mass *m* moving in a plane in a central force field. Write its Lagrangian in plane polar co-ordinates. Write the equations of motion and obtain the differential equation of the orbit.
9. Explain precessional motion.
10. Explain how a satellite in a circular orbit of radius *a* around the earth is send into an elliptical orbit around the earth with a distance of closest approach *a*.
11. What are transfer orbits? Explain briefly the steps involved in sending a space probe from earth to an outer planet.
12. Explain the working of communication satellites.
13. How did Kepler's laws pave the way for Newton to develop his law of gravitation?

PROBLEMS

1. A particle of mass *m* is observed to move in an orbit given by $r = kq$, where *k* is a constant. Determine the form of the force.
2. A particle describes the path $r = ke^{\alpha\theta}$, where *k* and *a* are constants, in a central force field. If the mass of the particle is *m*, find the law of force.
3. For a satellite in an elliptical orbit, the value of perigee and apogee distances from the centre of force are denoted by r_{min} and r_{max}, respectively. Show that the eccentricity $\epsilon = (r_{\text{max}} - r_{\text{min}})/(r_{\text{max}} + r_{\text{min}})$.
4. For a satellite in an elliptical orbit, the velocities at perigee and apogee points are u_p and u_a, respectively. Show that the eccentricity $\epsilon = (u_p - u_a)/(u_p + u_a)$.
5. A satellite in an elliptical orbit around the earth has the equation $\frac{9500}{r}\,\text{km} = 1 + 0.1\cos\theta$

Find the values of (i) the eccentricity (ii) the semi-major axis (iii) the semi-minor axis, and (iv) the period. Mass of the earth = 5.97 10^{24} kg, gravitational constant G = 6.67 10^{-11} Nm^2/kg^2.

6. A planet of mass *m* is moving around the sun in an elliptical orbit. If *M* is the mass of the sun, show that (i) energy $E = -GMm/(r_{min} + r_{max})$; (ii) period $T = 2pGM/(-2E/m)^{3/2}$.
7. The energy E_n and radius of the orbit r_n of hydrogen atom, according to quantum theory, are given by $E_n = -\frac{k^2 me^4}{2\hbar^2 n^2}$, $r_n = \frac{n^2\hbar^2}{kme^2}$, and $k = \frac{1}{4\pi\epsilon_0}$. Evaluate the frequency for the transition $n \to n-1$ for large quantum numbers and show that the results of quantum theory is consistent with Kepler's third law.
8. Show that the product of the maximum and minimum velocities of a particle moving in an elliptical orbit is $(2\pi a/T)^2$, where *a* is the value of the semi-major axis and *T* is the time period.
9. A particle of mass *M* moves in a central repulsive force field towards the centre of force with a velocity $\mathbf{u}_0$ and impact parameter *b*. If the force is $F(r) = k/r^3$, find the closest distance of approach to the centre of force.
10. A particle of mass *m* moves in an elliptical orbit about the centre of attractive force at one of its focus given by k/r^2, where *k* is a constant. If *a* is the semi-major axis, show that the speed **u** of the particle at any point of the orbit is $u = \sqrt{\frac{k}{m}}\left(\frac{2}{r} - \frac{1}{a}\right)^{1/2}$.
11. A particle of mass *m* moves in an elliptical orbit under the action of an inverse square central force. If *a* is the ratio of the velocity at perigee to that at apogee, show that the eccentricity $\epsilon = (\acute{a}-1)/(\acute{a}+1)$.
12. A particle moves in a circular orbit about the origin under the action of a central force $F(r) = -\frac{k}{r^3}$. If the potential energy is zero at infinity, find the total energy of the particle.
13. A particle of mass *m* describes the conic $r + r\epsilon\cos\theta = l$, where *l* and are constants. Find the force law.
14. A particle of mass *m* at the origin, acted upon by a central force, describe the curve $r = e^{-q}$, where r^3 are the plane polar co-ordinates. Show that the

magnitude of the force is inversely proportional to .

15. The orbital plane of an earth satellite coincides with that of the earth's equator. If it is at an altitude of 1000 km above the earth surface at perigee and 2000 km at the apogee, find (i) the eccentricity of the orbit, (ii) the semi-major and semi-minor axes, and (iii) the period of the satellite. The radius of the earth is 6380 km, the mass of the earth = 5.97×10^{24} kg, and the gravitational constant $G = 6.67\ 10^{-11}$ Nm^2/kg^2.
16. Show that at least three geostationary satellites are needed to cover all points on the equator of the earth.
17. A satellite is launched from the earth. At perigee, it is 636 km above the earth's surface and has a velocity of 9144 m/s. Calculate (i) the eccentricity, (ii) the apogee distance, and (iii) the velocity at the apogee. The radius of the earth = 6380 km, $G = 6.67\ 10^{-11}$ Nm^2/kg^2, and the mass of the earth = $5.97\ 10^{24}$ kg.

6

Hamiltonian Mechanics

In Lagrangian formalism, generalized coordinates (q_i's) and generalized velocities ($\dot{q}_i$'s) are used as independent coordinates to formulate dynamical problems which result in second order linear differential equations. In Hamilton's formalism, generalized coordinates and generalized momenta (p_i's) are used as basic variables to formulate problems. The formulation is mainly based on the Hamiltonian function of the system which is a function of q_i's and p_i's of the system. The resulting first order linear differential equations are easier to handle mathematically. Hamilton's formalism also serves as the basis for further developments such as Hamilton – Jacobi theory and quantum mechanics. Throughout this chapter, we shall assume that the systems are holonomic and the forces are derivable from a position-dependent potential.

6.1 THE HAMILTONIAN OF A SYSTEM

The Hamiltonian H of a system, defined by Eq. (3.69), is

$$H = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t) \tag{6.1}$$

where, as before, q stands for $q_1, q_2, \ldots, q_n$. Using the relation

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{6.2}$$

it is possible to express $\dot{q}_i$ in terms of p_i. When this is done, we can write

$$H = H(p, q, t) \qquad q = q_1, q_2, \ldots, q_n \qquad p = p_1, p_2, \ldots, p_n \tag{6.3}$$

That is, H is expressed as a function of the generalized coordinates, generalized momenta and time. In Lagrangian formalism, the configuration space is spanned by the *n* generalized coordinates. Here, the *q*'s and *p*'s are treated in the same way and the involved space is called the **phase space**. It is a space of 2*n* variables q_1, q_2,...,q_n, p_1, p_2,..., p_n. Every point in the space represents both the position and momenta of all particles in the system.

As already pointed out, in general, *H* need not represent the total energy of the system. However, if the transformation equations connecting the cartesian and generalized coordinates do not depend on time explicitly, *H* is equal to the total energy of the system.

6.2 HAMILTON'S EQUATIONS OF MOTION

Hamilton's equations of motion can be derived in the following different ways: (i) From the Hamiltonian of the system (ii) From the variational principle.

In this section we shall derive them from the Hamiltonian of a system given by Eq. (6.1). Differentiating Eq. (6.1), we have

$$dH = \sum_i \left[p_i d\dot{q}_i + \dot{q}_i \, dp_i - \frac{\partial L}{\partial q_i} \, dq_i - \frac{\partial L}{\partial \dot{q}_i} \, d\dot{q}_i \right] - \frac{\partial L}{\partial t} \, dt \qquad (6.4)$$

Since $p_i = \left(\frac{\partial L}{\partial \dot{q}_i} \right)$, the first and fourth terms on the right side of Eq. (6.4) together vanish. Hence,

$$dH = \sum_i \left(\dot{q}_i dp_i - \frac{\partial L}{\partial q_i} \, dq_i \right) - \frac{\partial L}{\partial t} dt \qquad (6.5)$$

Taking the differential of H in Eq. (6.3)

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \qquad (6.6)$$

Comparing Eqs. (6.5) and (6.6) and using Eq. (3.53), we get

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \; i = 1, 2, \ldots, n \qquad (6.7)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \; i = 1, 2, \ldots, n \qquad (6.8)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (6.9)$$

Equations (6.7) and (6.8) are **Hamilton's equations of motion**. They are also called the **canonical equations of motion**. They constitute a set of 2*n* first order differential equations replacing the *n* second order differential equations of Lagrange.

Hamilton's equations are applicable to holonomic conservative systems. If part of the forces acting on the system is not conservative, Lagrange's equations take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad (6.10)$$

where Q_i represents the forces not arising from a potential and L contains the conservative forces. Replacing $(\partial L/\partial \dot{q}_i)$ by p_i in Eq. (6.10), we have

$$\dot{p}_i = \frac{\partial L}{\partial q_i} + Q_i \qquad (6.11)$$

In such cases, Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (6.12)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i \qquad (6.13)$$

6.3 HAMILTON'S EQUATIONS FROM VARIATIONAL PRINCIPLE

Hamilton's variational principle stated in Eq. (4.2) is

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0 \qquad (6.14)$$

Here $q_i(t)$ and hence $\dot{q}_i(t)$ is to be varied such that

$$\delta q_i\,(t_1) = \delta q_i\,(t_2) = 0 \qquad (6.15)$$

which refers to paths in configuration space. In Hamilton's formalism, the integral I has to be evaluated over the trajectory of the system point in phase space, and the varied paths must be in the neighbourhood of this phase space trajectory. Therefore, to make the principle applicable to phase space trajectories, we have to express the integrand of the integral I as a function of the independent coordinates p and q and their time derivatives. This can be achieved only by replacing L in Eq. (6.14) using Eq. (6.1). We then get

$$\delta I = \delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(p, q, t) \right] dt = 0 \qquad (6.16)$$

where $q(t)$ is varied subject to $\delta q_i(t_1) = \delta q_i(t_2) = 0$ and $p_i(t)$ is varied without any end-point restriction. Since the original variational principle is modified to suit phase space, it is known as **modified Hamilton's principle**. Carrying out the variations in Eq. (6.16) we have

$$\int_{t_1}^{t_2} \sum_i \left(p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt = 0 \qquad (6.17)$$

We now integrate the first term in Eq. (6.17) by parts

$$\int_{t_1}^{t_2} p_i \delta \dot{q}_i \, dt = \int_{t_1}^{t_2} p_i \delta \frac{d}{dt} q_i \, dt = \int_{t_1}^{t_2} p_i \frac{d}{dt} \delta q_i \, dt$$

$$= [p_i \delta q_i]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_i \delta q_i \, dt$$

The integrated term vanishes at the end-points t_1 and t_2 and therefore

$$\int_{t_1}^{t_2} p_i \delta \dot{q}_i \, dt = - \int_{t_1}^{t_2} \dot{p}_i \delta q_i \, dt \qquad (6.18)$$

Substituting Eq. (6.18) in Eq. (6.17), we have

$$\int_{t_1}^{t_2} \sum_i \left[-\left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i + \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i \right] dt = 0 \qquad (6.19)$$

Since the modified Hamilton's principle is a variational principle in phase space, the *dq*'s and *dp*'s are arbitrary and therefore the coefficients of dq_i and dp_i in Eq. (6.19) must vanish separately. Hence,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \ldots, n \qquad (6.20)$$

Thus, Hamilton’s principle gives an independent method for obtaining Hamilton’s equations of motion without a prior Lagrangian formulation.

6.4 INTEGRALS OF HAMILTON'S EQUATIONS

Energy Integral

Hamiltonian H is a function of the generalized coordinates q_i, the generalized momenta p_i and time: $H = H(q, p, t)$ Differentiating with respect to time

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) + \frac{\partial H}{\partial t} \qquad (6.21)$$

If H does not depend on time explicitly, $(\partial H / \partial t) = 0.$ Replacing $\dot{q}_i$ and $\dot{p}_i$ using Hamilton’s equations,

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0 \qquad (6.22)$$

Hence,

$H(q, p) = \text{constant} = h$ (6.23) That is, H is conserved and the quantity h is called **Jacobi’s integral of motion**. If the holonomic constraints are time-independent and the potential is velocity-independent as shown in Eq. (3. 71), the Hamiltonian H is the total energy of the system.

$H = E = h$ (6.24) Integrals Associated with Cyclic Coordinates In Section 3.8, we defined a cyclic or ignorable coordinate as one that does not appear explicitly in the Lagrangian of a system. If the coordinate q_i is not appearing in the Lagrangian, $(\partial L / \partial q_i) = 0$ and

then $$\frac{\partial H}{\partial q_i} = \frac{\partial}{\partial q_i} \left(\sum_i p_i \dot{q}_i - L \right) = -\frac{\partial L}{\partial q_i} = 0 \qquad (6.25)$$

Hence, it will not be appearing in the Hamiltonian also. Combining the above equation with Hamilton’s equation, Eq. (6.8), we have

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \qquad (6.26)$$

$$p_i = \text{constant} = b \qquad (6.27)$$

That is, the momentum conjugate to a generalized coordinate which is cyclic is conserved.

Now, if we have a system in which the coordinates q_1, q_2,..., q_i are cyclic, then the Lagrangian of the system is of the form

$$L = L\,(q_{i+1}, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \qquad (6.28)$$

However, the Hamiltonian will be of the form

$$H = H\,(q_{i+1}, \ldots, q_n, b_1, b_2, \ldots, b_i, p_{i+1}, \ldots, p_n, t) \qquad (6.29)$$

When *i* cyclic coordinates are present in a system, in Lagrangian formalism the problem is still one of *n* degrees of freedom, whereas in Hamilton's formalism it is one of $(n - i)$ degrees of freedom. This is because even if q_j is absent in the Lagrangian, we have the equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad (6.30)$$

6.5 CANONICAL TRANSFORMATIONS

The transformation of one set of coordinates q_i to another set Q_i by transformation equations of the type $Q_i = Q_i\,(q_1, q_2, \ldots, q_n, t)$ (6.31) is called **point transformation** or **contact transformation**. What we have been doing in earlier chapters are transformations of this type. In Hamilton's formalism, the momenta are also independent variables on the same level as the generalized coordinates. Therefore, it is appropriate to have a more general type of transformation that involves both generalized coordinates and momenta. Considerable advantages will be there if the equations of motion are simpler in the new set of variables (Q, P) than in the original set (q, p). If all the coordinates are made cyclic by a transformation, the solutions will be much simpler. When there is a transformation from the original set (q, p) to the new set (Q, P), a corresponding change in the Hamiltonian $H\,(q, p, t)$ to a new Hamiltonian $K\,(Q, P, t)$ is expected. The transformation equations for the (q, p) to (Q, P) set are $Q_i = Q_i\,(q, p, t)$ and $P_i = P_i\,(q, p, t)$ (6.32) The (q, p) set obeys Hamilton's canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{6.33}$$

We are interested only in those transformations that are governed by Hamilton's canonical equations

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \text{and} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{6.34}$$

Such transformations are called **canonical transformations.**

The original set of variables satisfy the modified Hamilton's principle, given in Eq. (6.16):

$$\delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(q, p, t) \right] dt = 0 \tag{6.35}$$

The new set of variables (Q, P) should also hold the modified Hamilton's principle:

$$\delta \int_{t_1}^{t_2} \left[\sum_i P_i \dot{Q}_i - K(Q, P, t) \right] dt = 0 \tag{6.36}$$

The simultaneous validity of Eqs. (6.35) and (6.36) means that their integrands must be either equal or connected by a relation of the type

$$\sum_i p_i \dot{q}_i - H(q, p, t) = \alpha \left[\sum_i P_i \dot{Q}_i - K(Q, P, t) \right] + \frac{dF}{dt} \tag{6.37}$$

Here a is a constant independent of coordinates, momenta and time. This a is related to a simple type of scale transformation and therefore it is always possible to set $a = 1$. F is a function of the coordinates, momenta and time. The total time derivative of F in Eq. (6.37) will not contribute to the modified Hamilton's principle since

$$\delta\int_{t_1}^{t_2}\frac{dF}{dt}\,dt=\delta\int_{t_1}^{t_2}dF=\delta\left[\,F(t_2)-F(t_1)\right]=\delta F(t_2)-\delta F(t_1)=0$$

In the above, we have used the condition that the variation at the end-points is zero. This leads to

$$\sum_i p_i\dot{q}_i-H(q,\,p,t)=\sum_i P_i\dot{Q}_i-K(Q,P,t)+\frac{dF}{dt} \qquad (6.38)$$

The left hand side of Eq. (6.38) is a function of original coordinates and momenta, and the first two terms on the right hand side depends only on the (*Q*, *P*) set. Hence, in general, *F* must be a function of the original and new variables in order for a transformation to be effected. They are 4*n* in all. Of these 4*n*, only 2*n* are independent as the 4*n* variables are connected by the 2*n* equations of constraints given by Eq. (6.32). Hence, the function *F* can be written in 4 forms: F_1 (*q*, *Q*, *t*), F_2 (*q*, *P*, *t*), F_3 (*p*, *Q*, *t*) and F_4 (*p*, *P*, *t*). The problem in question will dictate which form is to be selected. Next we consider these 4 types in detail.

Type 1 – F_1 (q, Q, t): When the function *F* is of this form, Eq. (6.38) can be written as

$$\sum_i p_i\dot{q}_i-\sum_i P_i\dot{Q}_i+\left(K-H\right)=\sum_i\left(\frac{\partial F_1}{\partial q_i}\dot{q}_i+\frac{\partial F_1}{\partial Q_i}\dot{Q}_i\right)+\frac{\partial F_1}{\partial t} \qquad (6.39)$$

Multiplying by *dt* and rearranging

$$\sum_i\left(p_i-\frac{\partial F_1}{\partial q_i}\right)dq_i-\sum_i\left(P_i-\frac{\partial F_1}{\partial Q_i}\right)dQ_i+\left(K-H-\frac{\partial F_1}{\partial t}\right)dt=0 \qquad (6.40)$$

Since the original and new coordinates are separately independent, Eq. (6.40) is valid only if the coefficients of dq_i and dQ_i separately vanish. Therefore, from Eq. (6.40) we have

$$p_i=\frac{\partial F_1}{\partial q_i}(q,Q,t) \qquad (6.41)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}(q, Q, t) \qquad (6.42)$$

$$K = H + \frac{\partial F_1}{\partial t} \qquad (6.43)$$

From Eq. (6.41) we can compute Q in terms of q and p if the arbitrary function F_1 is known: $Q = Q\,(q, p, t)$ (6.44) Using this value of Q in Eq. (6.42) we can compute P in terms of q and p: $P = P\,(q, p, t)$ (6.45) Eqs. (6.44) and (6.45) are the desired transformations from the original (q, p) to the new (Q, P) set. Eq. (6.43) gives the relation connecting the original and new Hamiltonians. Thus, we can express (Q, P) in terms of (q, p) only if the arbitrary function F_1 is known. Hence, F_1 is called the **generating function** of the transformation. If the generating function F_1 does not contain time explicitly, then $K = H$.

Type 2 – F_2 (q, P, t): Addition of the term $-\frac{d}{dt}\sum_i P_i Q_i$ to the right hand side of Eq. (6.38) will not affect the value since F_2 is arbitrary and

$$\delta\int_{t_1}^{t_2}\frac{d}{dt}\sum_i P_iQ_i\ dt = \delta\int_{t_1}^{t_2}\sum_i d\left(P_iQ_i\right) = \delta\left[\sum_i P_iQ_i\right]_{t_1}^{t_2} = 0$$

It follows immediately from Eq. (6.38).

$$\sum_i p_i\dot{q}_i - H = \sum_i P_i\dot{Q}_i - K + \sum_i\frac{\partial F_2}{\partial q_i}\dot{q}_i + \sum_i\frac{\partial F_2}{\partial P_i}\dot{P}_i$$
$$+\frac{\partial F_2}{\partial t} - \sum_i P_i\dot{Q}_i - \sum_i Q_i\dot{P}_i \tag{6.46}$$

As two terms on the right hand side together vanish, multiplying by dt and rearranging

$$\sum_i\left(p_i - \frac{\partial F_2}{\partial q_i}\right)dq_i + \sum_i\left(Q_i - \frac{\partial F_2}{\partial P_i}\right)dP_i + \left(K - H - \frac{\partial F_2}{\partial t}\right)dt = 0 \tag{6.47}$$

Since q_i's and P_i's are independent

$$p_i = \frac{\partial F_2}{\partial q_i}(q, P, t) \tag{6.48}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}(q, P, t) \tag{6.49}$$

$$K = H + \frac{\partial F_2}{\partial t} \tag{6.50}$$

From Eq. (6.48)

$$P_i = P_i\,(q, p, t) \tag{6.51}$$

Using this result in Eq. (6.49)

$$Q_i = Q_i\,(q, p, t) \tag{6.52}$$

Eqs. (6.51) and (6.52) are the required transformation equations.

Type 3 – $F_3(p, Q, t)$: Proceeding on similar lines and adding the term $\frac{d}{dt}\sum_i p_i q_i$ to the right hand side of Eq. (6.38) and simplifying

$$P_i = -\frac{\partial F_3}{\partial Q_i}(p, Q, t) \tag{6.53}$$

$$q_i = -\frac{\partial F_3}{\partial p_i}(p, Q, t) \tag{6.54}$$

$$K = H + \frac{\partial F_3}{\partial t} \tag{6.55}$$

Type 4 – $F_4(p, P, t)$: Adding

$$-\frac{d}{dt}\sum_i P_i Q_i + \frac{d}{dt}\sum_i p_i q_i$$

to the right hand side of Eq. (6.38) and simplifying

$$q_i = -\frac{\partial F_4}{\partial p_i}(p, P, t) \tag{6.56}$$

$$Q_i = \frac{\partial F_4}{\partial P_i}(p, P, t) \tag{6.57}$$

$$K = H + \frac{\partial F_4}{\partial t} \tag{6.58}$$

In all these transformations, t is unchanged and therefore it may be regarded as an independent parameter. However, in relativistic formalism this cannot be so as space and time are treated on an equal footing. Sometimes a suitable generating function does not conform to one of the 4 types discussed above. Different combinations of the 4 types may be needed in such cases. If the generating function does not contain time explicitly, $K = H$ and Eq. (6.38) reduces to

$$\sum_i \left(p_i dq_i - P_i dQ_i\right) = dF \tag{6.59}$$

Then the condition for a transformation to be canonical is that $\sum_i \left(p_i dq_i - P_i dQ_i\right)$ must be a perfect differential.

6.6 POISSON BRACKETS

Hamilton's equations of motion for $\dot{q}$ and $\dot{p}$ give the time evolution of the coordinates and momenta of a system in phase space. Using these equations, we can find the equation of motion for any function $F(q, p)$ in terms of what is known as *Poisson brackets*. They are similar to commutator brackets in quantum mechanics and provide a bridge between classical mechanics and quantum mechanics (see Section 6.12).

The Poisson bracket of any two functions $F(q, p, t)$ and $G(q, p, t)$ with respect to the canonical variables (q, p), written as $[F, G]_{q,\,p}$, is defined by

$$[F, G]_{q,\,p} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \qquad (6.60)$$

Fundamental Poisson Brackets

1. $$[q_j, q_k]_{q,\,p} = 0 \qquad (6.61)$$

$$[q_j, q_k] = \sum_i \left(\frac{\partial q_j}{\partial q_i} \frac{\partial q_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i} \frac{\partial q_k}{\partial q_i} \right) = 0$$

since $(\partial q_k / \partial p_i) = (\partial q_j / \partial p_i) = 0$

2. $$[p_j, p_k]_{q,\,p} = 0 \qquad (6.62)$$

$$[p_j, p_k]_{q,\,p} = \sum_i \left(\frac{\partial p_j}{\partial q_i} \frac{\partial p_k}{\partial p_i} - \frac{\partial p_j}{\partial p_i} \frac{\partial p_k}{\partial q_i} \right) = 0$$

3. $$[q_j, p_k]_{q,\,p} = \delta_{jk} \qquad (6.63)$$

$$[q_j, p_k]_{q,\,p} = \sum_i \left(\frac{\partial q_j}{\partial q_i} \frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i} \frac{\partial p_k}{\partial q_i} \right)$$

The second term inside the bracket is zero and $(\partial q_j / \partial q_i) = \delta_{ij}$, $(\partial p_k / \partial p_i) = \delta_{ik}$. Hence,

$$[q_j, p_k]_{q,\,p} = \delta_{jk}$$

The above three brackets are called the **fundamental Poisson brackets**.

Fundamental Properties of Poisson Brackets

Let *F*, *G*, *S* be functions

of canonical variables (q, p) and time. The following fundamental identities can be obtained from the definition given in Eq. (6.60).

1. $[F, F] = 0$ (6.64)

$$[F, F] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial F}{\partial q_i} \right) = 0$$

2. $[F, C] = 0$, where C is a constant (6.65)

3. $[F, G] = -[G, F]$ (6.66)

$$[F, G] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) = -\sum_i \left(\frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i} \right) = -[G, F]$$

4. $[F, G+S] = [F, G] + [F, S]$ (6.67)

$$[F, G+S] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial (G+S)}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial (G+S)}{\partial q_i} \right)$$

$$= \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) + \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial S}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial S}{\partial q_i} \right)$$

$$= [F, G] + [F, S]$$

5. $[F, GS] = [F, G]S + G[F, S]$ (6.68)

$$[F, GS] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial (GS)}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial (GS)}{\partial q_i} \right)$$

$$= \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} S + G \frac{\partial F}{\partial q_i} \frac{\partial S}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} S - G \frac{\partial F}{\partial p_i} \frac{\partial S}{\partial q_i} \right)$$

$$= \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) S + \sum_i G \left(\frac{\partial F}{\partial q_i} \frac{\partial S}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial S}{\partial q_i} \right)$$

$$= [F, G]S + G[F, S]$$

6. Another important property of the Poisson bracket is the **Jacobi identity** for any three functions:

$$[F,[G,S]]+[G,[S,F]]+[S,[F,G]]=0 \tag{6.69}$$

7. $$\frac{d}{dt}[F,G]=\left[\frac{dF}{dt},G\right]+\left[F,\frac{dG}{dt}\right] \tag{6.70}$$

$$\frac{d}{dt}[F,G]=\frac{d}{dt}\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$=\sum_i\left[\frac{d}{dt}\left(\frac{\partial F}{dq_i}\right)\frac{\partial G}{\partial p_i}+\frac{\partial F}{\partial q_i}\frac{d}{dt}\left(\frac{\partial G}{\partial p_i}\right)\right.$$

$$\left.-\frac{d}{dt}\left(\frac{\partial F}{\partial p_i}\right)\frac{\partial G}{\partial q_i}-\frac{\partial F}{\partial p_i}\frac{d}{dt}\left(\frac{\partial G}{\partial q_i}\right)\right]$$

$$=\sum_i\left[\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial p_i}-\frac{\partial}{\partial p_i}\left(\frac{dF}{dt}\right)\frac{\partial G}{\partial q_i}\right.$$

$$\left.+\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{dG}{dt}\right)-\frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{dG}{dt}\right)\right]$$

$$=\left[\frac{dF}{dt},G\right]+\left[F,\frac{dG}{dt}\right]$$

A pair of functions for which the Poisson bracket $[F, G] = 0$ are said to *commute* with each other.

Equations of Motion in Poisson Bracket Form Consider a function F which is a function of q's, p's and time t: $F = F(q, p, t)$

$$F = F(q, p, t)$$

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) + \frac{\partial F}{\partial t}$$

Replacing $\dot{q}_i$ and $\dot{p}_i$ using Hamilton's equations

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial F}{\partial t}$$

$$= [F, H] + \frac{\partial F}{\partial t} \qquad (6.71)$$

which is the equation of motion of F in terms of Poisson bracket. In Eq. (6.71),
H is the Hamiltonian of the system. If F is replaced by q_j and p_j , Eq. (6.71) gives, when q_j and p_j do not depend explicitly on t

$$\dot{q}_j = [q_j, H] \quad \text{and} \quad \dot{p}_j = [p_j, H] \qquad (6.72)$$

These two equations constitute the canonical equations of motion in Poisson bracket form.

6.7 POISSON BRACKET AND INTEGRALS OF MOTION

One of the important uses of Poisson brackets is finding the integrals of motion. Let us consider Eq. (6.71) again. For F to be an integral of motion

$$\frac{dF}{dt} = 0 \qquad \text{or} \qquad [F, H] + \frac{\partial F}{\partial t} = 0 \qquad (6.73)$$

If the integral of motion F does not contain t explicitly, Eq. (6.73) reduces to $[F, H] = 0$ (6.74) That is, when the integral of motion does not contain t explicitly, its Poisson bracket with the Hamiltonian of the system vanishes. Conversely, the Poisson brackets of constants of motion with the Hamiltonian H must be zero.

Another important property of Poisson brackets is **Poisson's theorem** which states that if $F(q, p, t)$ and $G(q, p, t)$ are two integrals of motion, then $[F, G]$ is also an integral of motion. That is, $[F, G]$ = constant (6.75) Since F and G are integrals of motion

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] = 0 \quad \text{or} \quad \frac{\partial F}{\partial t} = -[F, H] \tag{6.76}$$

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + [G, H] = 0 \quad \text{or} \quad \frac{\partial G}{\partial t} = -[G, H] \tag{6.77}$$

Next, let us consider the time derivative of the Poisson bracket $[F, G]$. From Eq. (6.71), we have

$$\frac{d}{dt}[F, G] = [[F, G], H] + \frac{\partial [F, G]}{\partial t}$$

Using Eq. (6.70)

$$\frac{d}{dt}[F, G] = [[F, G], H] + \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$$

Replacing $(\partial F/\partial t)$ by Eq. (6.76) and $(\partial G/\partial t)$ by Eq. (6.77)

$$\frac{d}{dt}[F, G] = [[F, G], H] + [[H, F], G] + [[G, H], F]$$

Using Jacobi's identity given by Eq. (6.69)

$$\frac{d}{dt}[F, G] = 0$$

$$[F, G] = \text{constant} \tag{6.78}$$

6.8 THE CANONICAL INVARIANCE OF POISSON BRACKET

Probably the most important property of Poisson bracket is that it is invariant under canonical transformation. This means that if (q, p) and (Q, P) are two canonically conjugate sets, then $[F, G]_{q, p} = [F, G]_{Q, P}$ (6.79) where F and G are any pair of functions of (q, p) or (Q, P). The (q, p) and (Q, P) sets are related by a canonical transformation of the type given in Eq. (6.32): $Q_i = Q_i(q, p, t)$ and $P_i = P_i(q, p, t)$ The

Poisson bracket of the functions F and G with respect to the (q, p) set is given by

$$[F, G]_{q,p} = \sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \tag{6.80}$$

Consider the function

$$G = G(Q, P)$$

$$\frac{\partial G}{\partial p_k} = \sum_i \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial p_k} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial p_k} \right) \tag{6.81}$$

$$\frac{\partial G}{\partial q_k} = \sum_i \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial q_k} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial q_k} \right) \tag{6.82}$$

Substituting Eqs. (6.81) and (6.82) in Eq. (6.80)

$$[F, G]_{q,p} = \sum_k \left[\frac{\partial F}{\partial q_k} \sum_i \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial p_k} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial p_k} \right) - \frac{\partial F}{\partial p_k} \sum_i \left(\frac{\partial G}{\partial Q_i} \frac{\partial Q_i}{\partial q_k} + \frac{\partial G}{\partial P_i} \frac{\partial P_i}{\partial q_k} \right) \right] \tag{6.83}$$

$$= \sum_k \sum_i \left(\frac{\partial G}{\partial Q_i} \left(\frac{\partial F}{\partial q_k} \frac{\partial Q_i}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial Q_i}{\partial q_k} \right) + \frac{\partial G}{\partial P_i} \left(\frac{\partial F}{\partial q_k} \frac{\partial P_i}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial P_i}{\partial q_k} \right) \right)$$

$$= \sum_i \left\{ \frac{\partial G}{\partial Q_i} [F, Q_i]_{q,p} + \frac{\partial G}{\partial P_i} [F, P_i]_{q,p} \right\} \tag{6.84}$$

Replacing F by Q_k and G by F

$$[Q_k, F]_{q,p} = \sum_i \left\{ \frac{\partial F}{\partial Q_i} [Q_k, Q_i]_{q,p} + \frac{\partial F}{\partial P_i} [Q_k, P_i]_{q,p} \right\} \tag{6.85}$$

Since $\qquad [Q_k, Q_i] = 0 \qquad$ and $\qquad [Q_k, P_i] = \delta_{ik}$

$$[Q_k, F]_{q,p} = \frac{\partial F}{\partial P_k} \quad \text{or} \quad [F, Q_i]_{q,p} = -\frac{\partial F}{\partial P_i} \tag{6.86}$$

In the same way from Eq. (6.84) we have

$$[P_k, F]_{q,p} = -\frac{\partial F}{\partial Q_k} \quad \text{or} \quad [F, P_i]_{q,p} = \frac{\partial F}{\partial Q_i} \tag{6.87}$$

Substituting Eqs. (6.86) and (6.87) in Eq. (6.84)

$$[F, G]_{q,p} = \sum_i \left(\frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial Q_i} \right)$$

$$= [F, G]_{Q,P}$$

Thus, Poisson brackets are invariant under canonical transformation.

Poisson bracket description of mechanics is invariant under a canonical transformation. Therefore, a canonical transformation can be defined as one that preserves the Poisson bracket description of mechanics. Hence, we can make the following important statement: *The fundamental Poisson brackets Eqs. (6.61) to (6.63) provide the most convenient way to decide whether a given transformation is canonical*.

6.9 LAGRANGE BRACKETS

In addition to Poisson bracket, other canonical invariants exist. One such invariant is the Lagrange bracket. As its applications are limited, we will not elaborate it except for the definition and certain properties.

The Lagrange bracket of any two functions $F(q, p)$ and $G(q, p)$ with respect to (q, p) variables, written as $\{F, G\}_{q,p}$, is defined as

$$\{F, G\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial F} \frac{\partial p_i}{\partial G} - \frac{\partial p_i}{\partial F} \frac{\partial q_i}{\partial G} \right) \tag{6.88}$$

The Lagrange brackets are invariant under canonical transformations. That is, $\{F, G\}_{q,\, p} = \{F, G\}_{Q,\, P}$ (6.89) Hence, the subscripts (q, p) or (Q, P) may be dropped. From Eq. (6.88)

$$\{F, G\} = -\sum_i \left(\frac{\partial q_i}{\partial G} \frac{\partial p_i}{\partial F} - \frac{\partial p_i}{\partial G} \frac{\partial q_i}{\partial F} \right)$$

$$\{F, G\} = -\{G, F\} \tag{6.90}$$

If we take $F = q_k$ and $G = q_l$, from Eq. (6.88)

$$\{q_k, q_l\} = \sum_i \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_i}{\partial q_l} - \frac{\partial p_i}{\partial q_k} \frac{\partial q_i}{\partial q_l} \right)$$

$$\{q_k, q_l\} = 0 \tag{6.91}$$

In the same way

$$\{p_k, p_l\} = 0 \tag{6.92}$$

Taking $F = q_k$ and $G = p_l$

$$\{q_k, p_l\} = \sum_i \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_i}{\partial p_l} - \frac{\partial p_i}{\partial q_k} \frac{\partial q_i}{\partial p_l} \right) = \sum_i \frac{\partial q_i}{\partial q_k} \frac{\partial p_i}{\partial p_l}$$

$$\{q_k, p_l\} = \delta_{ik}\, \delta_{il} = \delta_{kl} \tag{6.93}$$

Equations (6.91) to (6.93) are called the *fundamental Lagrange brackets*.

The definitions of Poisson and Lagrange brackets clearly indicate some kind of inverse relationship between the two. The relation between the two is given by

$$\sum_{l=1}^{2n} \{F_l, F_i\} \left[F_l, F_j \right] = \delta_{ij} \tag{6.94}$$

Lagrange brackets do not obey Jacobi's identity.

6.10 D-VARIATION

The δ-variation that we considered in Section 4.1 refers to the variation in a quantity at the same instant of time. The varied path in configuration space always terminates at the end-points t_1 and t_2 such that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The Δ-variation, a more general type of variation of the path of the system, is one in which time as well as position co-ordinates vary in the configuration space. At the end-points of the path, the position co-ordinates are all kept fixed while the time co-ordinate may change. Fig. 6.1 illustrates the Δ-variation of a co-ordinate q_i in the configuration space.

Let ABC be the actual path and $A'\ B'\ C'$ the varied path. The end-points of the path A and C take the positions A' and C' after time Δt, such that there is no change in position co-ordinates, i.e., $\Delta q_i\ (1) = \Delta q_i\ (2) = 0$. The point B on the actual path now goes over to the point B' on the varied path such that

$$q_i' = q_i + \Delta q_i = q_i + \delta q_i + \dot{q}_i \Delta t \tag{6.95}$$

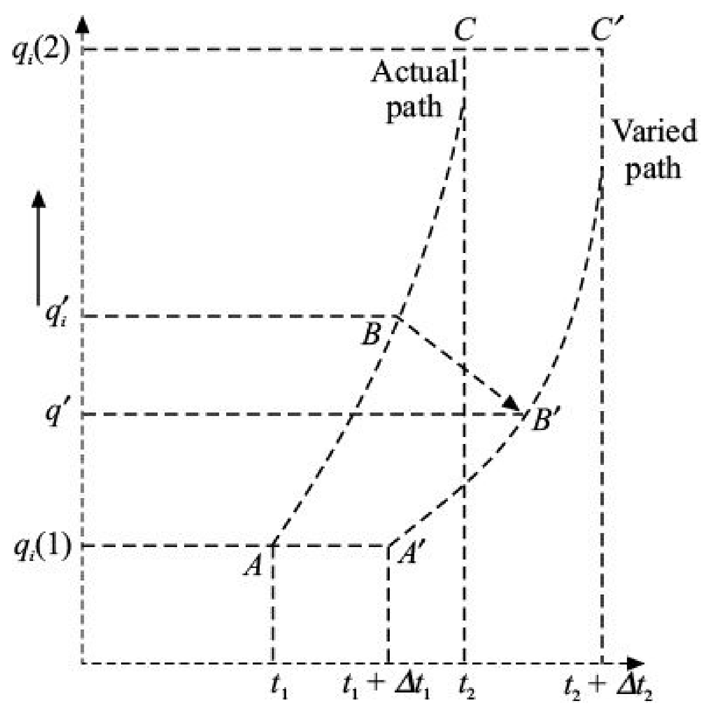


Fig. 6.1 Illustration of *D*-variation in configuration space.

The Δ-variation of any function $f = f(q, \dot{q}, t)$ is given by

$$\Delta f = \sum_i \left(\frac{\partial f}{\partial q_i} \Delta q_i + \frac{\partial f}{\partial \dot{q}_i} \Delta \dot{q}_i \right) + \frac{\partial f}{\partial t} \Delta t$$

Using Eq. (6.95)

$$\Delta f = \sum_i \frac{\partial f}{\partial q_i} \left(\delta q_i + \dot{q}_i \Delta t \right) + \sum_i \frac{\partial f}{\partial \dot{q}_i} \left(\delta \dot{q}_i + \ddot{q}_i \Delta t \right) + \frac{\partial f}{\partial t} \Delta t$$

Rearranging

$$\Delta f = \sum_i \left(\frac{\partial f}{\partial q_i} \delta q_i + \frac{\partial f}{\partial \dot{q}_i} \delta \dot{q}_i \right) + \left[\sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial f}{\partial t} \right] \Delta t$$

$$= \delta f + \Delta t \frac{df}{dt} \tag{6.96}$$

Thus, the δ and Δ operations are connected by the relation

$$\Delta = \delta + \Delta t \frac{d}{dt} \tag{6.97}$$

6.11 THE PRINCIPLE OF LEAST ACTION

The principle of least action is another variational principle associated with the Hamiltonian formulation. It involves the type of *D*-variation discussed in Section 6.10. To prove the principle of least action, consider the action integral

$$I = \int_{t_1}^{t_2} L\,dt \tag{6.98}$$

The Δ-variation of I is written using Eq. (6.97):

$$\begin{aligned}\Delta I &= \Delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \delta L\,dt + \Delta t \int_{t_1}^{t_2} \frac{d}{dt}\left(L\,dt\right) \\ &= \int_{t_1}^{t_2} \delta L\,dt + \Delta t \int_{t_1}^{t_2} dL \\ &= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i \right) dt + \left[L \Delta t \right]_{t_1}^{t_2}\end{aligned} \tag{6.99}$$

According to Lagrange's equation

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \tag{6.100}$$

An interchange in the order of differentiation gives

$$\delta \dot{q}_i = \frac{d}{dt} \delta q_i \tag{6.101}$$

Using Eqs. (6.100) and (6.101), the part in the parenthesis of the first term in Eq. (6.99) is

$$\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i = \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = \frac{d}{dt}(p_i\,\delta q_i) \tag{6.102}$$

Applying Eq. (6.97) for the co-ordinate q_i

$$\Delta q_i = \delta q_i + \Delta t\frac{d}{dt}q_i$$

$$\delta q_i = \Delta q_i - \Delta t \dot{q}_i$$

Multiplying by p_i

$$p_i \delta q_i = p_i \Delta q_i - p_i \Delta t\, \dot{q}_i \tag{6.103}$$

Substituting this value of $p_i\,\delta q_i$ in Eq. (6.102), we have

$$\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i = \frac{d}{dt}(p_i \Delta q_i) - \frac{d}{dt}(p_i\,\Delta t\,\dot{q}_i) \tag{6.104}$$

Combining Eqs. (6.99) and (6.104)

$$\Delta I = \Delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_i\left[\frac{d}{dt}(p_i\Delta q_i) - \frac{d}{dt}(p_i\Delta t\dot{q}_i)\right]dt + [L\Delta t]_{t_1}^{t_2}$$

$$= \sum_i [p_i\ \Delta q_i]_{t_1}^{t_2} - [p_i\dot{q}_i\Delta t]_{t_1}^{t_2} + [L\Delta t]_{t_1}^{t_2} \tag{6.105}$$

As $\Delta q_i = 0$ at the end-points, the first term on the right of Eq. (6.105) is zero. Hence,

$$\Delta \int_{t_1}^{t_2} L dt = \left[\left(L - \sum_i p_i \dot{q}_i \right) \Delta t \right]_{t_1}^{t_2} \tag{6.106}$$

Since $H = \sum_i p_i \dot{q}_i - L$, Eq. (6.106) reduces to

$$\Delta \int_{t_1}^{t_2} L\, dt = -\left[H\, \Delta t \right]_{t_1}^{t_2} \tag{6.107}$$

Restricting to systems for which $H = E$ $\left(\text{conservative and} \frac{\delta H}{\delta t} = 0 \right)$ or $\Delta H = 0$, we have

$$\Delta \int_{t_1}^{t_2} H\, dt = \int_{t_1}^{t_2} \Delta H\, dt + \int_{t_1}^{t_1} H \Delta (dt) = \int_{t_1}^{t_2} H d(\Delta t)$$

$$= \left[H\, \Delta t \right]_{t_1}^{t_2} \tag{6.108}$$

Combining Eqs. (6.107) and (6.108)

$$\Delta \int_{t_1}^{t_2} L\, dt = -\Delta \int_{t_1}^{t_2} H\, dt \quad \text{or} \quad \Delta \int_{t_1}^{t_2} (L + H)\, dt = 0$$

$$\Delta \int_{t_1}^{t_2} \sum_i \dot{q}_i p_i\, dt = 0 \tag{6.109}$$

which is the **principle of least action.**

Different Forms of Least Action Principle The principle of least action can be expressed in different forms. If the transformation equations do not depend on time explicitly, then the kinetic energy is a quadratic function of the generalized velocities. In such a case from
Eq. (3.47) we have

$$\sum_i p_i \dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = 2T \qquad (6.110)$$

Equation (6.109) now reduces to

$$\Delta \int_{t_1}^{t_2} T\, dt = 0 \qquad (6.111)$$

This is another form of the principle of least action. Further, if there is no external force on the system, T and H are conserved and the principle of least action takes the form

$$\Delta \int_{t_1}^{t_2} dt = 0 \quad \text{or} \quad \Delta(t_2 - t_1) = 0 \qquad (6.112a)$$

$$t_2 - t_1 = \text{an extremum} \qquad (6.112b)$$

That is, of all paths possible between two points that are consistent with the conservation of energy, the system moves along the path for which the time of transit is the least. In this form, the principle is similar to **Fermat's principle** in geometrical optics, which states that a light ray travels between two points along such a path that the time taken is the least.

Again, when the transformation equations do not involve time, the kinetic energy is given by Eq. (3.45):

$$T = \sum_j \sum_k a_{jk}\, \dot{q}_j\, \dot{q}_k \tag{6.113}$$

A configuration space for which the a_{ij} coefficients form the metric tensor can be constructed. The element of path length $d\rho$ in this space is defined by

$$(d\rho)^2 = \sum_j \sum_k a_{jk}\, dq_j\, dq_k \tag{6.114}$$

$$\left(\frac{d\rho}{dt}\right)^2 = \sum_j \sum_k a_{jk}\, \dot{q}_j\, \dot{q}_k \tag{6.115}$$

From Eqs. (6.113) and (6.115)

$$T = \left(\frac{d\rho}{dt}\right)^2 \quad \text{or} \quad dt = \frac{d\rho}{\sqrt{T}} \tag{6.116}$$

Equation (6.116) helps us to change the variable in Eq. (6.111) and the principle of least action takes the form

Equation (6.116) helps us to change the variable in Eq. (6.111) and the principle of least action takes the form

$$\Delta \int_{t_1}^{t_2} T dt = \Delta \int_{t_1}^{t_2} \sqrt{T}\; d\rho = 0 \tag{6.117}$$

For conservative systems, $H = T + V$. Consequently, Eq. (6.117) becomes

$$\Delta \int_{t_1}^{t_2} \sqrt{H - V(q)}\; d\rho = 0 \tag{6.118}$$

Eq. (6.118) is often referred to as **Jacobi's form of least action principle**. It now refers to the path of the system in a curvilinear configuration space characterized by the metric tensor with elements a_{jk}.

6.12 POISSON BRACKETS AND QUANTUM MECHANICS

In classical physics, the state of a system at a given time t is specified by equations of motion. The dynamical variables occurring in these equations are the position coordinate **r**, linear momentum **p**, angular momentum **L**, and so on. In quantum mechanics, commutators replace the Poisson brackets. The commutator of dynamical variables A and B, written as $[A, B]$, is defined as $[A, B] = AB - BA$ (6.119) The properties of commutators are similar to those of Poisson brackets.

In quantum mechanics, dynamical variables of classical physics are replaced by operators. The operators in quantum mechanics are derived from the Poisson bracket of the corresponding pair of classical variables according to the rule $[\hat{q}, \hat{r}] = i\hbar \{q, r\}$ (6.120) where $\hat{q}$ and $\hat{r}$ are the operators selected for the dynamical variables q and r, and $\{q, r\}$ is the Poisson bracket of q and r. As an example, consider the dynamical variables x and p_x. The Poisson bracket of x with p_x $\{x, p_x\} = 1$

The operators selected for x and p_x are $\hat{x}$ and $\hat{p}_x$ respectively. Then, by the above rule, the commutator of $\hat{x}$ with $\hat{p}_x$ is given by $[\hat{x}, \hat{p}_x] = i\hbar\{x, p_x\} = i\hbar$ (6.121) Operators associated with dynamical variables and their commutators play a crucial role in quantum mechanics.

WORKED EXAMPLES

Example 6.1 Obtain Hamilton's equations for a simple pendulum. Hence, obtain an expression for its period.

Solution: Figure 6.2 illustrates the pendulum.

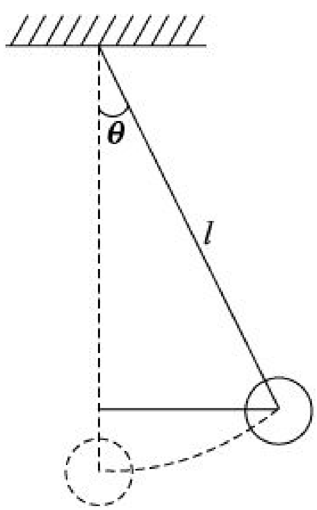


Fig. 6.2 Simple pendulum We use q as the generalized coordinate. For evaluating potential energy, the

energy corresponding to the mean position is taken as zero. The velocity of the bob $\mathbf{v} = l\dot{\theta}$.

Kinetic energy $\quad T = \frac{1}{2} m\, l^2 \dot{\theta}^2$

Potential energy $\quad V = mgl\,(1 - \cos\theta)$

$$L = T - V = \frac{1}{2} ml^2\dot{\theta}^2 - mgl\left(1 - \cos\theta\right) \tag{i}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \text{or} \quad \dot{\theta} = \frac{p_\theta}{ml^2}$$

Hamiltonian $H(\theta, p_\theta) = \dot{\theta} p_\theta - L$

$$= \frac{1}{2ml^2}\, p_\theta^2 + mgl\left(1 - \cos\theta\right) \tag{ii}$$

Hamilton's equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\,mgl\sin\theta \tag{iii}$$

$$\ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = -\,\frac{g\sin\theta}{l}$$

Since θ is small, $\sin\,\theta \cong \theta$ and the above equation reduces to

$$\ddot{\theta} = \frac{-\,g\theta}{l} \tag{iv}$$

The motion is simple harmonic and the period T is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{v}$$

Example 6.2 Obtain Hamilton's equations for a particle of mass *m* moving in a plane about a fixed point by an inverse square force $-k/r^2$. Hence, (i) obtain the radial equation of motion; (ii) show that the angular momentum is constant.

Solution: In plane polar coordinates, the kinetic energy *T* and potential energy *V* are given by

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad V = -k/r \tag{i}$$

Lagrangian
$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r} \tag{ii}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{or} \quad \dot{r} = \frac{p_r}{m} \tag{iii}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \text{or} \quad \dot{\theta} = \frac{p_\theta}{mr^2} \tag{iv}$$

Hamiltonian
$$H = \sum_i p_i \dot{q}_i - L = p_r\dot{r} + p_\theta\dot{\theta} - L$$

Substituting $\dot{r}$ and $\dot{\theta}$ and simplifying

$$H = \frac{1}{2m} p_r^2 + \frac{1}{2mr^2} p_\theta^2 - \frac{k}{r} \tag{v}$$

Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \tag{vi}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{vii}$$

(i) From the two equations in Eq. (vi), we have

$$\dot{p}_r = m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \tag{viii}$$

which is the radial equation of motion.

which is the radial equation of motion.

(ii) The second equation of Eq. (vii) gives p_q = constant (ix) which is the law of conservation of angular momentum, since $p_\theta = mr^2\dot{\theta}$, the angular momentum.

Example 6.3 A mass *m* is suspended by a massless spring of spring constant *k*. The suspension point is pulled upwards with constant acceleration a_0. Find the Hamiltonian of the system, Hamilton's equations of motion and the equation of motion.

Solution: Let the vertical be the z-axis. As the acceleration due to gravity is downwards, taking the net acceleration as $(g - a_0)$.

Potential energy $\quad V = \frac{1}{2}kz^2 + m(g - a_0)z$

Kinetic energy $\quad T = \frac{1}{2}m\dot{z}^2$

$$L = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kz^2 - m(g - a_0)z \qquad \text{(i)}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \text{or} \quad \dot{z} = \frac{p_z}{m}$$

$$H = p_z\,\dot{z} - L = \frac{p_z^2}{2m} + \frac{1}{2}kz^2 + m(g - a_0)z \qquad \text{(ii)}$$

Hamilton's equations are

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \text{(iii)}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz - m(g - a_0) \qquad \text{(iv)}$$

The equation of motion is

$$\ddot{z} = \frac{1}{m}\dot{p}_z = \frac{1}{m}[-kz - m(g - a_0)]$$

$$m\ddot{z} = -kz - m(g - a_0) \qquad \text{(v)}$$

Example 6.4 A bead of mass m slides on a frictionless wire under the influence of gravity (see Fig. 6.3). The shape of the wire is parabolic and it rotates about the z-axis with constant angular velocity w. Taking $z^2 = ar$ as the equation of the parabola, obtain the Hamiltonian of the system. Is $H = E$?

Solution: Figure 6.3 illustrates the motion of the bead.

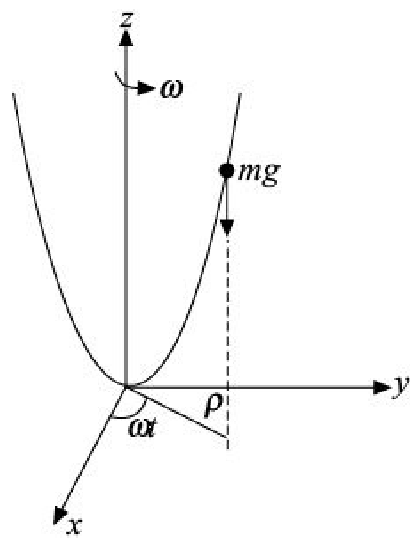


Fig. 6.3 Bead sliding down on a wire.

The wire rotates with constant angular velocity *w* and therefore we may write

$$x = \rho\cos\omega t \quad \text{and} \quad y = \rho\sin\omega t \tag{i}$$

$$z^2 = a\rho \tag{ii}$$

$$2z\dot{z} = a\dot{\rho} \quad \text{or} \quad \dot{\rho} = \frac{2}{a}z\dot{z}$$

$$\dot{x} = \dot{\rho}\cos\omega t - \rho\omega\sin\omega t$$

$$\dot{y} = \dot{\rho}\sin\omega t + \rho\omega\cos\omega t$$

$$\dot{x}^2 + \dot{y}^2 = \dot{\rho}^2 + \rho^2\,\omega^2 = \frac{4z^2\dot{z}^2}{a^2} + \frac{z^4\omega^2}{a^2}$$

Kinetic energy $\quad T = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$= \frac{1}{2}m\left(\frac{4z^2}{a^2}\dot{z}^2 + \frac{z^4\omega^2}{a^2} + \dot{z}^2\right)$$

$$= \frac{1}{2}\frac{m}{a^2}[\dot{z}^2\,(a^2 + 4z^2) + \omega^2 z^4]$$

$$L = \frac{m}{2a^2}[\dot{z}^2(a^2 + 4z^2) + \omega^2 z^4] - mgz \tag{iii}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = \frac{m\dot{z}}{a^2}(a^2 + 4z^2) \quad \dot{z} = \frac{p_z a^2}{m(a^2 + 4z^2)}$$

$$H = \sum_i p_i\,\dot{q}_i - L = \dot{z}\,p_z - L \tag{iv}$$

Substituting the value of $\dot{z}$ and simplifying

$$H = \frac{a^2 p_z^2}{2m(a^2 + 4z^2)} - \frac{m\omega^2}{2a^2}z^4 + mgz \tag{v}$$

Total energy $\quad E = T + V$

$$= \frac{p_z^2\;a^2}{2m\;(a^2 + 4z^2)} + \frac{m\omega^2 z^4}{2a^2} + mgz \tag{vi}$$

It is evident from Eqs. (v) and (vi) that $E \neq H$.

Example 6.5 A particle of mass *m* moves in three dimensions under the action of a central conservative force with potential energy *V*(*r*).Then (i) Find the Hamiltonian function in spherical polar coordinates; (ii) Show that *f* is an ignorable coordinate; (iii) Obtain Hamilton's equations of motion; and (iv) Express the quantity $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{or} \quad \dot{r} = \frac{p_r}{m}$ in terms of generalized momenta.

Solution: (i) Kinetic energy $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2)$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{or} \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \text{or} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta\dot{\phi} \quad \text{or} \quad \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}$$

$$H = \sum_i p_i\dot{q}_i - L = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$$

Substituting the values of $\dot{r}, \dot{\theta}$ and $\dot{\phi}$, we have

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right] + V(r)$$

(ii) The coordinate *f* is not appearing in the Hamiltonian. Hence, it is an ignorable coordinate.

(iii) Hamilton's canonical equations will be six in number as there are three generalized coordinates. They are

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{dV(r)}{dr} \qquad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{mr^2}\frac{p_\phi^2 \cos\theta}{\sin^3\theta} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2\theta}$$

(iv) $$l^2 = m^2 r^4 \left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = m^2 r^4 \left(\frac{p_\theta^2}{m^2 r^4} + \frac{\sin^2\theta\, p_\phi^2}{m^2 r^4 \sin^4\theta}\right)$$

$$= p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$$

Example 6.6 Obtain the Hamiltonian of a charged particle in an electromagnetic field.

Solution: The Lagrangian of a charged particle in an electromagnetic field is given by Eq. (3.82)

$$L = \frac{1}{2}m\mathbf{v}^2 - U \qquad U = q\phi - q(\mathbf{A}.\mathbf{v})$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi + q(A_x\dot{x} + A_y\dot{y} + A_z\dot{z})$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \qquad \dot{x} = \frac{p_x - qA_x}{m}$$

$$p_y = m\dot{y} + qA_y \qquad \dot{y} = \frac{p_y - qA_y}{m}$$

$$p_z = m\dot{z} + qA_z \qquad \dot{z} = \frac{p_z - qA_z}{m}$$

Hamiltonian

$$H = \sum_i p_i\dot{q}_i - L = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m\mathbf{v}^2 + q\phi - q(\mathbf{A}\cdot\mathbf{v})$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q(A_x\dot{x} + A_y\dot{y} + A_z\dot{z}) - \frac{1}{2}m\mathbf{v}^2 + q\phi - q(\mathbf{A}\cdot\mathbf{v})$$

$$= \frac{1}{2}m\mathbf{v}^2 + q\phi = \frac{m^2}{2m}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q\phi$$

$$= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi$$

Example 6.7 Find the canonical transformation generated by the generating function $F_1 = q_i\, Q_i$.

Solution: When the generating function is a function of q_i and Q_i, Eq. (6.41)

$$p_i = \frac{\partial F_1}{\partial q_i} = \frac{\partial}{\partial q_i}(q_i\, Q_i) = Q_i$$

gives Using Eq. (6.42)

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -\frac{\partial}{\partial Q_i}(q_i\, Q_i) = -q_i$$

In effect, the transformation interchanges the coordinates and momenta, except for the negative sign in the second one. The new coordinates are the old momenta and the new momenta are the old coordinates. In other words, the distinction between them is one of nomenclature. Thus, Hamilton's formalism treats coordinates and momenta on an equal footing.

Example 6.8 Solve the problem of simple harmonic oscillator in one dimension by effecting a canonical transformation.

Solution: The Hamiltonian of the oscillator $H = \frac{p^2}{2m} + \frac{kq^2}{2}$ (i) is obtained in terms of the coordinate q and the momentum p. To make the solution simpler, let us have a transformation in which the new coordinate Q is cyclic; then P will be a constant of motion. Consider the generating function

$$F_1 = C q^2 \cot Q, \text{ where } C \text{ is a constant.} \tag{ii}$$

Eqs. (6.41) and (6.42) give

$$p = \frac{\partial F_1}{\partial q} = 2Cq \cot Q \tag{iii}$$

$$P = -\frac{\partial F_1}{\partial Q} = Cq^2 \operatorname{cosec}^2 Q \tag{iv}$$

$$q = \sqrt{\frac{P}{C}} \sin Q \tag{v}$$

Substituting this value of q in Eq. (iii)

$$p = 2\sqrt{CP} \cos Q \tag{vi}$$

Substituting the values of q and p in Eq. (i), we have

$$K = H = \frac{kP}{2C} \left(\sin^2 Q + \frac{4C^2}{mk} \cos^2 Q \right) \tag{vii}$$

Writing the constant $C = \sqrt{mk}/2$

$$K = H = P\sqrt{\frac{k}{m}} \tag{viii}$$

As the new Hamiltonian does not contain the new co-ordinate Q, the new conjugate momentum P is a constant of motion:

$$P = \alpha \text{ (constant)} \tag{ix}$$

Hamilton's equation for Q gives

$$\dot{Q} = \frac{\partial K}{\partial P} = \sqrt{\frac{k}{m}}$$

Integrating

$$Q = \sqrt{\frac{k}{m}}\, t + \beta \qquad \beta = \text{constant} \tag{x}$$

Substituting Eq. (ix) in Eq. (vi) and Eq. (x) in Eq. (v), we have

$$p = \text{constant} \times \cos\left(\sqrt{\frac{k}{m}}\, t + \beta\right) \tag{xi}$$

$$q = \text{constant} \times \sin\left(\sqrt{\frac{k}{m}}\, t + \beta\right) \tag{xii}$$

It is true that the use of canonical transformation has not simplified the harmonic oscillator problem. It is given here to illustrate the procedure.

Example 6.9 Show that the following transformation is canonical.

$Q = \sqrt{2q}\, e^{\alpha} \cos p \qquad P = \sqrt{2q}\, e^{-\alpha} \sin p$ *a* is a constant

Solution: The differential $dQ = e^{\alpha}\left(-\sqrt{2q}\,\sin p\,dp + \frac{\cos p}{\sqrt{2q}}\,dq\right)$

$$PdQ - pdq = \sqrt{2q}\,e^{-\alpha}\sin p\,e^{\alpha}\left(-\sqrt{2q}\,\sin p\,dp + \frac{\cos p}{\sqrt{2q}}\,dq\right) - pdq$$

$$= -2q\sin^2 p\,dp + \sin p\cos p\,dq - pdq$$

$$= -2q\sin^2 p\,dp + \left(\frac{\sin 2p}{2} - p\right)dq$$

$$\frac{d}{dp}\left(\frac{\sin 2p}{2} - p\right) = \cos 2p - 1 = -2\sin^2 p$$

$$d\left(\frac{\sin 2p}{2} - p\right) = -2\sin^2 p\,dp$$

Consequently,

$$PdQ - pdq = q\,d\left(\frac{\sin 2p}{2} - p\right) + \left(\frac{\sin 2p}{2} - p\right)dq$$

$$= d\left[q\left(\frac{\sin 2p}{2} - p\right)\right]$$

Since, the right hand side is a perfect differential, the transformation is canonical.

Example 6.10 Show that the transformation

$p = m\omega q\cot Q$ and $P = \dfrac{m\omega q^2}{2\sin^2 Q}$

is canonical, and obtain the generator of the transformation.

Solution: From the given data, $\cot Q = \frac{p}{m\omega q}$

$$-\operatorname{cosec}^2 Q \, dQ = \frac{1}{m\omega}\left(\frac{dp}{q} - \frac{pdq}{q^2}\right)$$

$$dQ = \frac{\sin^2 Q}{m\omega}\left(\frac{pdq}{q^2} - \frac{dp}{q}\right)$$

$$PdQ - pdq = \frac{m\omega q^2}{2\sin^2 Q}\,\frac{\sin^2 Q}{m\omega}\left(\frac{pdq}{q^2} - \frac{dp}{q}\right) - pdq$$

$$= \frac{pdq}{2} - \frac{qdp}{2} - pdq = -\frac{1}{2}(pdq + qdp) = -d\left(\frac{pq}{2}\right)$$

Hence, the transformation is canonical. The generator of the transformation can easily be written using Eq. (6.59):

$$F = \frac{1}{2}pq = \frac{1}{2}q\,m\omega q \cot Q = \frac{1}{2}m\omega q^2 \cot Q$$

Example 6.11 Using the Poisson bracket, show that the transformation $q = \sqrt{2P}\sin Q \qquad p = \sqrt{2P}\cos Q$

is canonical.

Solution: From the definition of Poisson bracket, it is obvious that
$[Q, Q] = 0$ and $[P, P] = 0$.

From the given data, we have

$$\tan Q = \frac{q}{p} \qquad \text{and} \qquad 2P = q^2 + p^2$$

$$\sec^2 Q \frac{\partial Q}{\partial q} = \frac{1}{p} \qquad \text{or} \qquad \frac{\partial Q}{\partial q} = \frac{\cos^2 Q}{p}$$

$$\sec^2 Q \frac{\partial Q}{\partial p} = -\frac{q}{p^2} \qquad \text{or} \qquad \frac{\partial Q}{\partial p} = -\frac{q}{p^2} \cos^2 Q$$

$$\frac{\partial P}{\partial q} = q \qquad \frac{\partial P}{\partial p} = p$$

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \frac{\cos^2 Q}{p} p + \frac{q^2}{p^2}\cos^2 Q = \cos^2 Q\left(1 + \frac{q^2}{p^2}\right)$$

$$= \cos^2 Q\left(1 + \tan^2 Q\right) = 1$$

Hence, the transformation is canonical.

Example 6.12 Find the Poisson bracket of $[L_x, L_y]$, where L_x and L_y are angular momentum components.

Solution: Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

$$L_x = yp_z - zp_y \quad L_y = zp_x - xp_z \quad L_z = xp_y - yp_x$$

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$$

$$= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$

Consider the bracket $[yp_z, xp_z]$.

$$[yp_z, xp_z] = [y, x]\, p_z p_z + y[p_z, x]p_z + x[y, p_z]\, p_z + xy[p_z, p_z] = 0$$

since all the fundamental brackets involved are zero. In the same way

$$[zp_y, zp_x] = 0$$

Next we shall consider the Poisson bracket $[yp_z, zp_x]$.

$$[yp_z, zp_x] = [y, z]\, p_z p_x + y[p_z, z]\, p_x + z[y, p_x]\, p_z + zy[p_z, p_x]$$

$$= 0 + y(-1)\; p_x + 0 + 0 = -\, yp_x$$

In the same way

$$[zp_y, xp_z] = x(+1)\; p_y = xp_y$$

Substituting all the brackets

$$[L_x, L_y] = xp_y - yp_x = L_z$$

Proceeding on the same lines, we can show that

$$[L_y, L_z] = L_x \quad \text{and} \quad [L_z, L_x] = L_y$$

Note: In general, $[L_i, L_j] = L_k$, where i, j and k are taken in cyclic order. Let us introduce a symbol ϵ_{ijk} with the following meaning:

(i) $\epsilon_{ijk} = 0$ if two indices are equal.

$\epsilon_{iii} = \epsilon_{iik} = \epsilon_{iji} = 0$

(ii) $\epsilon_{ijk} = 1$, if i, j, k are distinct and in cyclic order.

$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = 1$

(iii) $\epsilon_{ijk} = -1$, if i, j, k are distinct and not in cyclic order.

$\epsilon_{ikj} = \epsilon_{jik} = \epsilon_{kji} = -1$

The implication of the above result is that no two components of angular momentum can simultaneously act as conjugate momenta, since conjugate momenta must obey the relation $[p_i, p_j] = 0$. Only one angular momentum component can be chosen as a generalized coordinate in any particular system of reference.

Example 6.13 For what values of *a* and *b* $Q = q^{\alpha} \cos \beta p \qquad P = q^{\alpha} \sin \beta p$ represent a canonical transformation. Also find the generator of the transformation.

Solution: $\dfrac{\partial Q}{\partial q} = \alpha q^{\alpha-1} \cos \beta p \qquad \dfrac{\partial P}{\partial q} = \alpha q^{\alpha-1} \sin \beta p$

$$\frac{\partial Q}{\partial p} = -\beta q^{\alpha} \sin \beta p \qquad \frac{\partial P}{\partial p} = \beta q^{\alpha} \cos \beta p$$

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \alpha\beta q^{2\alpha-1} \cos^2 \beta p + \alpha\beta q^{2\alpha-1} \sin^2 \beta p = \alpha\beta q^{2\alpha-1}$$

For the transformation to be canonical, this must be equal to 1. Hence,

$$\alpha\beta q^{2\alpha-1} = 1$$

The right hand side is dimensionless. Hence, $q^{2\alpha-1}$ is dimensionless. Therefore,

$$2\alpha - 1 = 0 \quad \text{or} \quad \alpha = \frac{1}{2}$$

$$\alpha\beta = 1 \quad \text{or} \quad \beta = 2$$

With $\qquad \alpha = \dfrac{1}{2} \quad \text{and} \quad \beta = 2$

$$Q = q^{1/2} \cos 2p \quad \text{or} \quad q = \frac{Q^2}{\cos^2 2p}$$

Since q is a function of p and Q, the generating function must be of the type

$$F = F_3(p, Q) \quad \text{and} \quad q = -\frac{\partial F_3}{\partial p} \quad \text{or} \quad F_3 = -\int q\,dp$$

$$F_3 = -\int \frac{Q^2}{\cos^2 2p}\,dp = -\frac{1}{2} Q^2 \tan 2p$$

Example 6.14 Show that the transformation $Q = ln\left(\frac{1}{q}\sin p\right)$ and $P = q\cot p$ is canonical. Also obtain the generating function for the transformation.

Solution: From the given data

$$dQ = \frac{q}{\sin p}\left[-\frac{\sin p}{q^2}dq + \frac{\cos p}{q}dp\right] = -\frac{dq}{q} + \cot p\, dp$$

$$P\,dQ - pdq = q\cot p\left(-\frac{dq}{q} + \cot p\,dp\right) - p\,dq$$

$$= -[(p + \cot p)\,dq - q\cot^2 p\,dp]$$

$$= -d[q(p + \cot p)]$$

since

$$d\,[q(p + \cot p)] = dq(p + \cot p) + q(dp - \operatorname{cosec}^2 p\,dp)$$

$$= (p + \cot p)\,dq - q\cot^2 dp$$

Hence, the transformation is canonical. From the given data

$$\cot p = \frac{P}{q} \quad \text{or} \quad p = \cot^{-1}\left(\frac{P}{q}\right)$$

$$PdQ - pdq = -\,d\left[q\left(\cot^{-1}\frac{P}{q} + \frac{P}{q}\right)\right]$$

Hence, the generator of the transformation is given by

$$F(q, P) = \left(q\cot^{-1}\frac{P}{q} + P\right)$$

Example 6.15 Obtain Hamilton's equations for the projectile motion of a particle of mass m in the gravitational field. Hence, show that the cyclic coordinate in it is proportional to the time of flight if the point of projection is the origin.

Solution: The motion of the projectile is in the two-dimensional *xy*-plane. Coordinates x and y can be taken as the generalized coordinates. Potential energy $V = mgy$, where y is the height above the earth.

Lagrangian $$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$H = \sum_i p_i \dot{q}_i - L = p_x \dot{x} + p_y \dot{y} - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{1}{2m}(p_x^2 + p_y^2) + mgy$$

$$= \frac{1}{2m}(p_x^2 + p_y^2) + mgy$$

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{P_y}{m}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = -mg$$

From the expressions for L and H, it is obvious that x is the cyclic co-ordinate. From the first and the third equations

$$m\dot{x} = p_x = \text{constant} \quad \text{or} \quad \dot{x} = \frac{\text{constant}}{m}$$

Integrating

$$x = At + B \qquad A \text{ and } B \text{ are constants}$$

Since the point of projection is the origin, $B = 0$. Hence, the required result.

REVIEW QUESTIONS

1. Define the Hamiltonian of a system. Under what conditions, is it the total energy of the system?
2. A coordinate which is cyclic in Lagrangian formalism is also cyclic in Hamilton's formalism. Substantiate.
3. State and explain Hamilton's modified principle.
4. Distinguish between configuration space and phase space.
5. Explain the salient features of Hamilton's formalism of mechanics.
6. What is a canonical transformation?
7. Distinguish between point transformation and canonical transformation.

8. Define Poisson bracket and state its important properties.
9. Obtain the equation of motion of a dynamical variable $F(q, p, t)$ in terms of the Poisson bracket.
10. State and explain Jacobi identity.
11. If F (q, p, t) and $G(q, p, t)$ are two integrals of motion, show that the Poisson bracket $[F, G]$ is also an integral of motion.
12. If q_j and p_j do not depend on time explicitly, show that $\dot{q}_j = [q_j, H]$ and $\dot{p}_j = [p_j, H]$.
13. F, G and S are functions of (q, p, t), prove that $[FG, S] = F[G, S] + [F, S]G$
14. Show that $\frac{d}{dt}[F, G] = \left[\frac{dF}{dt}, G\right] + \left[F, \frac{dG}{dt}\right]$
15. Show that the Poisson brackets of constants of motion with the Hamiltonian is zero.
16. The fundamental Poisson brackets provide the most convenient way to decide whether a given transformation is canonical. Discuss.
17. Explain the principle of least action, bringing out clearly the type of variation involved.
18. How does the principle of least action lead to Fermat's principle in geometrical optics?
19. Express the principle of least action in Jacobi's form.
20. Poisson brackets provide a bridge between classical and quantum mechanics. Substantiate.

PROBLEMS

1. A mass m is suspended by a massless spring of spring constant k. If the mass executes simple harmonic motion, write its Hamiltonian. Hence, obtain its equation of motion.
2. A mass m is suspended by a massless spring of spring constant k. If the mass executes simple harmonic motion, determine its phase-space trajectories.
3. A particle of mass m is constrained to move on the surface of a cylinder of radius a. It is subjected to an attractive force towards the origin which is proportional to the distance of the particle from the origin. Obtain its Hamiltonian and Hamilton's equations of motion.
4. A particle of mass m and charge q moves in a plane in a central field potential $V(r)$. A constant magnetic field **B** is applied perpendicular to the plane of

rotation. Find the Hamiltonian in a fixed frame of the observer.

5. Find the canonical transformation generated by the following generating functions: (i) $F(q, p) = \sum_{i=1}^{n} P_i q_i$ and (ii) $F(q, P) = -\sum_{i} P_i q_i$.

6. Is the transformation $Q = \frac{1}{p}$ and $P = qp^2$ canonical? If canonical, find the generator of the transformation.

7. Prove that the transformation defined by the equations $P = 2(1 + q^{1/2} \cos p)\, q^{1/2} \sin p, Q = ln(1 + q^{1/2} \cos p)$ is canonical. Also find the generating function.

8. The transformation equations between two sets of coordinates are $Q = (q^2 + p^2)/2$ and $P = -\tan^{-1}(q/p)$. Show that the transformation is canonical.

9. Given the canonical transformations $Q = (q^2 + p^2)/2$, $P = -\tan^{-1}(q/p)$. Evaluate $[Q, P]$ and show $[H, [Q, P]] = 0$.

10. If the Hamiltonian of a system is $H = \frac{p^2}{2} - \frac{1}{2q^2}$, show that $F = \frac{pq}{2} - Ht$ is a constant of motion.

11. Obtain the Poisson bracket of (i) $[L_x, p_y]$ and (ii) $[x, L_y]$, where L_x and L_y are the x and y components of angular momentum.

12. Show that the Poisson bracket $[L^2, L_i] = 0$ where $i = x, y, z$.

13. Using Poisson bracket, show that the following transformation is canonical: $Q = \sqrt{e^{-2q} - p^2}$, $P = \cos^{-1}(pe^q)$. Also find the generator of the transformation.

14. Consider the motion of a free particle of mass m. A constant of its motion is $F = x - pt/m$. Show that $(\partial F/\partial t) = [H, F]$.

15. Find the condition to be satisfied by a, b, c and d so that the transformation $Q = aq + bp$, $P = cq + dp$ is canonical.

16. A mass m is suspended by a massless spring having spring constant k and unstretched length r_0. Find Hamilton's equations and the equations of motion of the mass m if the mass is allowed to swing as a simple pendulum.

17. Use Hamilton's method to obtain the equation of motion for a spherical pendulum and show that the angular momentum about a vertical axis through

the point of support is a constant of motion.

18. If all the coordinates of a dynamical system of n degrees of freedom are ignorable, prove that the problem is completely integrable.

7

Hamilton-Jacobi Theory

The canonical transformation which we discussed in the previous chapter leads us to the Hamilton-Jacobi theory which is an equivalent formulation of classical mechanics. Two slightly different approaches are available. In one, the procedure is to find a canonical transformation from the original set (*q*, *p*) to a new set of variables (*Q*, *P*) which makes all coordinates cyclic. Then the new momenta will be constants in time. The second approach is to effect a canonical transformation such that the new Hamiltonian *K* (*Q*, *P*, *t*) is zero, then each $\dot{Q}_i$ and $\dot{P}_i$ is zero. It means that all *Q*'s and *P*'s will be constants of motion. The second approach is a more general one and we shall discuss such a transformation in this chapter.

7.1 HAMILTON–JACOBI EQUATION

Consider a dynamical system with Hamiltonian *H* (*q*, *p*, *t*). If a canonical transformation is made from the (*q*, *p*, *t*) set to (*Q*, *P*, *t*) set with the transformed Hamiltonian *K* = 0, Hamilton's equations will be

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \tag{7.1}$$

$$\dot{P}_i = \frac{-\partial K}{\partial Q_i} = 0 \tag{7.2}$$

As seen in Section 6.5, the original Hamiltonian *H* and the new Hamiltonian *K* are related by the equation

$$K = H + \frac{\partial F}{\partial t} \tag{7.3}$$

For *K* to be zero

$$H + \frac{\partial F}{\partial t} = 0 \tag{7.4}$$

where F is the generating function of the transformation.

It is convenient to take F as a function of the original coordinates q_i and the new constant momenta P_i and time t, which corresponds to F_2 (q, P, t) of Section 6.5. From Eq. (6.48)

$$p_i = \frac{\partial F_2}{\partial q_i} \tag{7.5}$$

With this value of p_i, Eq. (7.4) becomes

$$H\left(q_1,\ldots,q_n, \frac{\partial F_2}{\partial q_1}, \frac{\partial F_2}{\partial q_2}, \ldots, \frac{\partial F_2}{\partial q_n}, t\right) + \frac{\partial F_2}{\partial t} = 0 \tag{7.6}$$

Equation (7.6) is known as the **Hamilton Jacobi (H-J) equation**. It is the practice to denote the solution of Eq. (7.6) by S. The function S is called **Hamilton's principal function**. Replacing F_2 by S

$$H\left(q_1,\ldots,q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0$$ (7.6a) Equation (7.6a) is a first order differential equation in the (n + 1) variables q_1, q_2,..., q_n and t. Hence, the solution will have (n + 1) independent constants of integration. In the H-J equation, only partial derivatives of the type $(\partial S/\partial q_i)$ and $(\partial S/\partial t)$ appear. S as such does not appear in the equation. Therefore, if S is a solution, $S + a$, where a is a constant, is also a solution. From the (n + 1) constants $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$, we may choose $\alpha_{n+1} = \alpha$ as an additive constant. Hence, a complete solution of Eq. (7.6) can be written as $S = S(q_1, q_2, \ldots, q_n, \alpha_1, \alpha_2, \ldots, \alpha_n, t)$ (7.7) Comparison of S in Eq. (7.7) with F_2 in Eq. (6.48) suggests that the constants of integration are the new momenta

$P_i = a_i$ (7.8) The n transformation equations denoted by Eq. (6.48) now take the form $p_i = \dfrac{\partial S(q, \pm, t)}{\partial q_i}$ (7.9) The other n transformation equations given by Eq. (6.49) gives the new constant coordinates $Q_i = \beta_i = \dfrac{\partial S}{\partial \alpha_i}(q, \alpha, t)$

(7.10) By calculating the right side of Eq. (7.10) at $t = t_0$ the constant b 's can be obtained in terms of the initial values q_i, q_2,..., q_n. From Eq. (7.10) we can get

q_i in terms of (a, b, t): $q_i = q_i(\alpha, \beta, t)$ (7.11)

After the differentiation in Eq. (7.9), substitution of Eq. (7.11) for q gives the momenta p_i in terms of (a, b, t):

$$p_i = p_i(\alpha, \beta, t) \tag{7.12}$$

Hamilton's principal function is thus a generator of a canonical transformation to constant coordinates and momenta.

Physical Significance of *S*

From Eq. (7.7)

From Eq. (7.7)

$$\frac{dS}{dt} = \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial S}{\partial \alpha_i}\dot{\alpha}_i + \frac{\partial S}{\partial t}$$

Since the α's are constants, $\dot{\alpha}_i = 0$. Using Eq. (7.9)

$$\frac{dS}{dt} = \sum_i p_i \dot{q}_i + \frac{\partial S}{\partial t} \tag{7.13}$$

Using Eq. (7.6a) to replace $(\partial S / \partial t)$

$$\frac{dS}{dt} = \sum_i p_i \dot{q}_i - H = L \tag{7.14}$$

Integrating

$$S = \int L dt + \text{constant} \tag{7.15}$$

That is, Hamilton's principal function differs at most from the indefinite time integral of the Lagrangian only by a constant.

7.2 HAMILTON'S CHARACTERISTIC FUNCTION

In systems where the time-dependent part of Hamilton's principal function S could be separated out, the integration of the H-J equation is straightforward. Such a separation of variables is always possible whenever the original Hamiltonian does not depend on time explicitly. In such cases, Eq. (7.6a) reduces to $H\left(q_1, q_2, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}\right) + \frac{\partial S}{\partial t} = 0$ (7.16)

The first term involves only the *q*'s whereas the second term depends on time. Hence, the time variation can be separated by assuming a solution for *S* of the type

$$S(q, \alpha, t) = W(q, \alpha) - \alpha_1 t \tag{7.17}$$

Since

$$\frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i} \qquad \text{and} \qquad \frac{\partial S}{\partial t} = -\alpha_1$$

substituting the trial solution Eq. (7.17) in Eq. (7.16), we get

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1 \tag{7.18}$$

This equation does not involve time. By virtue of Eq. (7.18), the constant of integration a_1 appearing in *S* is equal to the constant value of *H*, which is the total energy *E* if *H* does not depend on time explicitly. The function *W*(*q*, *a*) is called **Hamilton's characteristic function**. Since *W* does not involve time, the original and new Hamiltonians are equal and hence $K = a_1$. Replacing a_1 by *E* in Eq. (7.18) $H\left(q, \frac{\partial W}{\partial q}\right) = E$ (7.19)

Equation (7.19) is the Hamilton-Jacobi equation for Hamilton's characteristic function. From Eqs. (7.10) and (7.17)

$$Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} = \begin{cases} \dfrac{\partial W}{\partial \alpha_i} & \text{if } i \neq 1 \\[2ex] \dfrac{\partial W}{\partial \alpha_i} - t & \text{if } i = 1 \end{cases} \tag{7.20}$$

$$Q_i = \beta_i + t = \frac{\partial W}{\partial \alpha_i} \quad \text{if} \quad i = 1 \tag{7.21a}$$

$$Q_i = \beta_i = \frac{\partial W}{\partial \alpha_i} \quad \text{if} \quad i \neq 1 \tag{7.21b}$$

Thus, Q_1 is the only coordinate which is not a constant of motion.

The physical significance of Hamilton's characteristic function can easily be obtained. We have

$$W = W(q, \alpha)$$

$$\frac{dW}{dt} = \sum_i \frac{\partial W}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial W}{\partial \alpha_i}\dot{\alpha}_i = \sum_i \frac{\partial W}{\partial q_i}\dot{q}_i$$

Since $$p_i = \frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i}$$

$$\frac{dW}{dt} = \sum_i p_i \dot{q}_i$$

Integrating

$$W = \int \sum_i p_i \dot{q}_i dt = \int \sum_i p_i dq_i \qquad (7.22)$$

The integral in Eq. (7.22) can be considered an abbreviated action integral.

7.3 HARMONIC OSCILLATOR IN THE H-J METHOD

What we discussed in the H-J theory can be applied for solving the motion of a one-dimensional harmonic oscillator. The Hamiltonian of the system is

$$H = \frac{1}{2m} p^2 + \frac{1}{2} kq^2 = E \qquad \omega^2 = \frac{k}{m} \tag{7.23}$$

Since $p = \partial S / \partial q$, the H-J equation, Eq. (7.6a), is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} kq^2 + \frac{\partial S}{\partial t} = 0 \tag{7.24}$$

The function S has explicit dependence on t only in the last term and therefore a solution of S can be written as

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \tag{7.25}$$

where α is the integration constant. From Eq. (7.25)

$$\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} \qquad \text{and} \qquad \frac{\partial S}{\partial t} = -\alpha \tag{7.26}$$

Substituting these values in Eq. (7.24), we get

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} kq^2 = \alpha \tag{7.27}$$

Since the left side of this equation is Hamiltonian, the constant *a* is simply the total energy *E* of the system. Equation (7.27) can be written as

$$\frac{\partial W}{\partial q} = \sqrt{2m\alpha - mkq^2} \tag{7.28}$$

$$dW = \sqrt{2m\alpha}\ \sqrt{1 - \frac{kq^2}{2\alpha}}\ dq$$

$$W = \sqrt{2m\alpha}\ \int \sqrt{1 - \frac{kq^2}{2\alpha}}\ dq \tag{7.29}$$

Substituting this value of W in Eq. (7.25)

$$S = \sqrt{2m\alpha}\int \sqrt{1 - \frac{kq^2}{2\alpha}}\ dq - \alpha t \tag{7.30}$$

Use of Eq. (7.10) gives

$$\beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\ \int \sqrt{1 - \frac{kq^2}{2\alpha}}\ dq + \sqrt{\frac{m}{2\alpha}}\int \frac{\left(kq^2/2\alpha\right)}{\sqrt{1 - \left(kq^2/2\alpha\right)}}\ dq - t$$

$$= \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1 - \left(kq^2/2\alpha\right)}} - t$$

The above integral is a standard one, and integration gives

$$t + \beta = \sqrt{\frac{m}{k}}\ \sin^{-1}\left(q\sqrt{\frac{k}{2\alpha}}\right) \tag{7.31}$$

Since $k = m\omega^2$, Eq. (7.31) reduces to

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\ \sin\omega(t + \beta) \tag{7.32}$$

The momentum p can be obtained using Eq. (7.9) or simply by identifying $p = m\dot{q}$

$$p = \sqrt{2m\alpha}\ \cos\omega(t + \beta) \tag{7.33}$$

The constants a and b can be related to the initial values of q_0 and p_0. If the particle is at rest at $t = 0$, $p = p_0$ and the particle is displaced from the equilibrium position by q_0. Squaring Eqs. (7.33) and (7.32) and adding

$$\frac{p^2}{2m\alpha}+\frac{q^2 m\omega^2}{2\alpha}=1$$

Substituting the initial conditions

$$\frac{q_0{}^2 m\omega^2}{2\alpha}=1 \qquad \text{or} \qquad \alpha=\frac{q_0{}^2 m\omega^2}{2} \tag{7.34}$$

Also, from Eq. (7.32) we have

$$q_0 = q_0 \sin\omega\beta \qquad \text{or} \qquad \sin\omega\beta = 1$$

$$\omega\beta=\frac{\pi}{2} \qquad \text{or} \qquad \beta=\frac{\pi}{2\omega} \tag{7.35}$$

Using Eq. (7.27) the constant α can easily be identified as the total energy E. From Eq. (7.6a)

$$\frac{\partial S}{\partial t}+H=0$$

Since the system is conservative, using Eq. (7.27)

$$\alpha = H = E \tag{7.35a}$$

Thus, Hamilton's principal function S is the generator of a canonical transformation to a new coordinate that measures the phase angle of the oscillation and to a new canonical momentum a identified as the total energy.

7.4 SEPARATION OF VARIABLES IN THE H-J EQUATION

The solution of differential equations in the H-J formalism is somewhat complicated. However, if the variables in the H-J equation are separable, the procedure becomes much simpler. In fact, it becomes a useful tool to solve problems only when such a separation is possible.

A variable q_j is said to be *separable* in H-J equation if Hamilton's principal function can be split into two additive parts, one of which depends only on the coordinate q_j and the constant momenta, and the other is completely independent of q_j. If q_1 is the separable coordinate, then the Hamiltonian must be such that

$$S(q_1 \ldots q_n, \alpha_1 \ldots \alpha_n, t) = S_1(q_1, \alpha_1 \ldots \alpha_n, t) + S'(q_2 \ldots q_n, \alpha_1 \ldots \alpha_n, t) \tag{7.36}$$

Then the H-J equation splits into two equations, one for S_1 and the other for S. If all the coordinates in a problem are separable, then the H-J equation is said to be *completely separable*. In such a case

$$S = \sum_i S_i\,(q_i, \alpha_1, \ldots, \alpha_n, t) \tag{7.37}$$

and the H-J equation splits into n equations of the type

$$H_i\left(q_i, \frac{\partial S_i}{\partial q_i}, \alpha_1, \ldots, \alpha_n\right) = \alpha_i \tag{7.38}$$

The constants a_i are referred to as the *separation constants*. It may be noted that each equation in Eq. (7.38) involves only one of the coordinates q_i and the corresponding partial derivative $(\partial S_i / \partial q_i)$. Therefore, one has to solve the equation only for the partial derivative and then integrate over the variable q_i.

In conservative mechanical systems, t is a separable variable in the H-J equation and a solution for S can be written in the form

$$S(q, \alpha, t) = S_0(\alpha, t) + W(q, \alpha) \tag{7.39}$$

Since H is not an explicit function of time, the H-J equation with this solution takes the form

$$H\left(q, \frac{\partial W}{\partial q}\right) + \frac{\partial S_0}{\partial t} = 0$$

$$H\left(q, \frac{\partial W}{\partial q}\right) = -\frac{\partial S_0}{\partial t} \tag{7.40}$$

In Eq. (7.40) the left hand side is a function of q alone and the right hand side is a function of t alone. This is possible only when each side is a constant. Then,

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1 \tag{7.41}$$

$$\frac{\partial S_0}{\partial t} = -\alpha_1 \tag{7.41a}$$

The solution of the second equation is

$$S_0 = -\alpha_1 t$$

Equation (7.41) is the H-J equation for *W*. This equation implies that the separation constant a_1 is the energy of the system.

If the coordinate q_1 is cyclic, then the conjugate momentum p_1 is a constant, say *g*. The H-J equation for *W* is then

$$H\left(q_2,\ldots,q_n,\gamma,\frac{\partial W}{\partial q_2},\ldots,\frac{\partial W}{\partial q_n}\right)=\alpha_1 \qquad (7.42)$$

Let the separated solution be of the form

$$W(q,\alpha)=W_1(q_1,\alpha)+W'(q_2\ldots q_n,\alpha) \qquad (7.43)$$

The H-J equation with this trial solution leads to two equations, one for W_1 and the other for *W*. The equation for *W* is of the same form as Eq. (7.42); the other one is

$$\gamma=\frac{\partial W_1}{\partial q_1} \qquad (7.44)$$

The solution of Eq. (7.44), except for an integration constant, gives

$$W_1=\gamma q_1 \qquad (7.44a)$$

The function *W* can now be written as

$$W=W'+\gamma q_1 \qquad (7.45)$$

If all the co-ordinates except one (say q_1) are cyclic, then the H-J equation can be completely separated by repeated application of the above procedure. The separated form for *W* is then

$$W=W_1(q_1,\alpha)+\sum_{i=2}^{n}\alpha_i\, q_i \qquad (7.46)$$

where W_1 is the solution of the reduced H-J equation.

$$H\left(q_1;\frac{\partial W_1}{\partial q_1},\alpha_2,\alpha_3,\ldots,\alpha_n\right)=\alpha_1 \qquad (7.47)$$

Solution of Eq. (7.47) can easily be obtained by solving for the partial derivative $(\partial W_1/\partial q_1)$ and then integrating over q_1.

7.5 CENTRAL FORCE PROBLEM IN PLANE POLAR COORDINATES

For an example of the ideas about separability developed in the previous section, we consider the H-J equation for a particle moving in a central force field in plane polar coordinates. The motion involves only two degrees of freedom. In plane polar coordinates, the Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r) \tag{7.48}$$

The Hamiltonian does not involve time and hence it is a constant of motion and equals E. The variable q is cyclic. Hence, the conjugate momentum p_q is a constant of motion. Hamilton's characteristic function is separable and is given by

Eq. (7.46)

$$W = W_1(r) + \alpha_\theta \theta \tag{7.49}$$

where α_θ is the constant angular momentum p_θ. Now

$$p_r = \frac{\partial W}{\partial r} = \frac{\partial W_1}{\partial r} \tag{7.50}$$

The H-J equation Eq. (7.18) can now be written as

$$\frac{1}{2m}\left(\frac{\partial W_1}{\partial r} \right)^2 + \frac{\alpha_\theta^2}{2mr^2} + V(r) = \alpha_1$$

$$\frac{\partial W_1}{\partial r} = \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\theta^2}{r^2}}$$

$$W_1 = \int \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\theta^2}{r^2}}\, dr \tag{7.51}$$

The function W is

$$W = \int \sqrt{2m(\alpha_1 - V) - \frac{\alpha_\theta^2}{r^2}}\, dr + \alpha_\theta \theta \tag{7.52}$$

With this form of the characteristic function, the transformation equations, Eqs. (7.21a) and (7.21b) become

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \int \frac{m dr}{\sqrt{2m(\alpha_1 - V) - \alpha_\theta^2 / r^2}} \tag{7.53}$$

and,

$$\beta_2 = \frac{\partial W}{\partial \alpha_\theta} = -\int \frac{\alpha_\theta dr}{r^2 \sqrt{2m(\alpha_1 - V) - \alpha_\theta^2 / r^2}} + \theta \tag{7.54}$$

Equation (7.53) gives t as a function of r, which agrees with the corresponding expression, Eq. (5.39), in central force motion with the energy a_1 written as E and the angular momentum a_q as L. Changing the variable of integration r by $u = 1/r$, Eq. (7.54) reduces to

$$\theta = \beta_2 - \int \frac{du}{\sqrt{\frac{2m}{\alpha_\theta^2}(\alpha_1 - V) - u^2}} \tag{7.54a}$$

This gives the orbit equation which agrees with the one in Eq. (5.50) with the constant b_2 as q_0.

7.6 ACTION-ANGLE VARIABLES

Periodic systems are very common in physics. Consider a conservative periodic system with one degree of freedom. The Hamiltonian is then $H(q, p) = a_1$ (7.55) Here a_1 is the energy E. Solving for the momentum p, we have $p = p(q, a_1)$ (7.56) For periodic motion, we now introduce a new variable J to replace a_1 as the transformed constant momentum. It is called the **action variable** or **phase integral**, defined as

$$J = \oint p\, dq \tag{7.57}$$

where the integration is over a complete period. The variable J has the dimensions of angular momentum. It follows from Eqs. (7.56) and (7.57) that J is a function of α_1. Hence, we can write

$$\alpha_1 = H = H(J) \tag{7.58}$$

Hamilton's characteristic function W can be written as

$$W = W(q, J) \tag{7.59}$$

The generalized co-ordinate conjugate to J is called the **angle variable** w. It is defined by the transformation equation

$$w = \frac{\partial W(q, J)}{\partial J} \tag{7.60}$$

In terms of w and J, Hamilton's equations are

$$\dot{w} = \frac{\partial H(J)}{\partial J} = \nu(J) \qquad \text{and} \qquad \dot{J} = -\frac{\partial H}{\partial w_i} \tag{7.61}$$

where $\nu(J)$ is a constant function of J only. Integration of the first of Eq. (7.61) gives $w = \nu t + \beta$ (7.62)

By solving Eq. (7.60), one can get q as a function of w and J. This results in combination with Eq. (7.62) gives the solution connecting q and time.

To get a physical interpretation for the constant *n*, consider a change in the angle variable w as q goes through a complete cycle of the periodic motion:

$$\Delta w = \oint \frac{\partial w}{\partial q}\, dq \tag{7.63}$$

Replacing *w* by Eq. (7.60)

$$\Delta\, w = \oint \frac{\partial^2 W}{\partial q \partial J}\, dq \tag{7.64}$$

Since *J* is a constant, the derivative with respect to *J* can be taken out. Therefore,

$$\Delta w = \frac{d}{dJ} \oint \frac{\partial W}{\partial q}\, dq = \frac{d}{dJ} \oint p dq \tag{7.65}$$

Since

$$\oint p\, dq = J$$

$$\Delta w = \frac{dJ}{dJ} = 1 \tag{7.66}$$

That is, the angle variable *w* changes by unity as *q* goes through a complete cycle. If τ is the period for a complete cycle, from Eq. (7.62)

$$\Delta w = \nu \Delta t = \nu\tau \tag{7.67}$$

Combining Eqs. (7.66) and (7.67)

$$\nu\tau = 1 \qquad \text{or} \qquad \nu = \frac{1}{\tau} \tag{7.68}$$

which means n is the frequency associated with the periodic motion of *q*. If the Hamiltonian is determined as a function of *J*, the frequency of the motion can be determined using Eq. (7.61).

Thus, one can get the frequency of periodic motion without solving the problem completely. This is done by finding the Hamiltonian as a function of *J* and then taking the derivative of *H* with respect to *J*. The variable *J* has the dimension of angular momentum and the coordinate conjugate to angular momentum is an angle; hence the name *angle variable*.

The application of the action-angle procedure to systems of more than one degree of freedom requires the concept of **multiply-periodic system**. In this case, the motion of the system is said to be *periodic* if the projection of the system point on each (q_i, p_i) plane is simply periodic.

7.7 HARMONIC OSCILLATOR IN ACTION-ANGLE VARIABLES

For an example of action-angle variables to find frequencies, let us consider the linear harmonic oscillator problem. The H-J equation of the harmonic oscillator is given by Eq. (7.27).

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}kq^2 = \alpha = E$$

$$p = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - mkq^2} \qquad k = m\omega^2 \tag{7.69}$$

The action variable

$$J = \oint p\,dq = \oint \sqrt{2m\alpha - mkq^2}\,dq$$

$$= \sqrt{2m\alpha} \quad \oint \sqrt{1 - \frac{kq^2}{2\alpha}}\,dq \tag{7.70}$$

Since p is real

$$\frac{2\alpha}{k} \geq q^2 \qquad \text{or} \qquad q \leq \pm\sqrt{\frac{2\alpha}{k}} \tag{7.71}$$

The extreme values of q are $-\sqrt{2\alpha/k}$ and $\sqrt{2\alpha/k}$. Let

$$q = \sqrt{\frac{2\alpha}{k}}\sin\theta \qquad dq = \sqrt{\frac{2\alpha}{k}}\cos\theta\,d\theta.$$

When $\quad q = -\sqrt{2\alpha/k}, \qquad \sin\theta = -1 \qquad \theta = -\pi/2$

When $\quad q = \sqrt{2\alpha/k}, \qquad \sin\theta = 1 \qquad \theta = \pi/2$

That is, when q changes from q_{min} to q_{max}, q changes from $-p/2$ to $p/2$. On the return, q changes from $p/2$ to $-p/2$. Hence, during one period, q changes from

$$J = \sqrt{2m\alpha}\ \sqrt{\frac{2\alpha}{k}}\ \int_0^{2\pi} \cos^2\theta\, d\theta$$

0 to 2*p*. Now

$$= 2\alpha \sqrt{\frac{m}{k}}\ \pi \qquad (7.72a)$$

$$\alpha = H = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ J \qquad (7.72b)$$

Frequency of oscillation

$$v = \frac{\partial H}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (7.73)$$

which is the usual expression for the frequency of a simple harmonic oscillator. From Eq. (7.62), the angle variable

$$w = vt + \beta = \frac{\omega}{2\pi}t + \beta$$

$$2\,\pi w = \omega(t + \beta') \qquad (7.74)$$

That is, if β' is suitably defined, the factor $\omega(t + \beta)$ in Eqs. (7.32) and (7.33) is the same as $2\pi w$. Hence, the solutions for q, Eq. (7.32), and p, Eq. (7.33), in action angle variables take the form

$$q = \sqrt{\frac{J}{\pi m\omega}}\ \sin 2\pi w \qquad (7.75a)$$

$$p = \sqrt{\frac{mJ\omega}{\pi}}\cos 2\pi w \qquad (7.75b)$$

Equations (7.75a) and (7.75b) relate the canonical variables (w, J) to the canonical variables (q, p).

7.8 KEPLER PROBLEM IN ACTION-ANGLE VARIABLES

Kepler's problem is a very general one, unlike the linear harmonic oscillator considered in the previous section. In the Kepler problem a planet of mass *m* moves around the sun in space. In spherical polar coordinates the Hamiltonian is

$$H = \frac{1}{2m}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) + V(r)$$

Since $V = -k/r$, in terms of canonical momenta

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{k}{r} \tag{7.76}$$

The co-ordinate ϕ is cyclic and therefore

$$p_\phi = \text{constant} \tag{7.77}$$

The H-J equation for the system is given by

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right] - \frac{k}{r} = E \tag{7.78}$$

Since the variables are separable

$$W(r, \theta, \phi) = W_1(r) + W_2(\theta) + W_3(\phi) \tag{7.79}$$

On substituting Eq. (7.79) in Eq. (7.78) and simplifying, Eq. (7.78) splits into the following three equations:

$$\left(\frac{dW_3}{d\phi}\right)^2 = \alpha_\phi^2 = \text{constant} \qquad \alpha_\phi = p_\phi \tag{7.80}$$

$$\left(\frac{dW_2}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 = \text{constant} \tag{7.81}$$

$$\left(\frac{dW_1}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m\left(E + \frac{k}{r}\right) \tag{7.82}$$

Integration of Eqs. (7.80), (7.81) and (7.82) gives the generating functions W_3, W_2 and W_1. The sum of them will give the generating function W.

We shall restrict our discussion to bound orbits, that is, those for which energy E is negative. In such a case, the motion will be periodic in (r,q, f) coordinates. Next we shall find the action-angle variables of the system.

We have

$$p_r = \frac{dW_1}{dr} \qquad p_\theta = \frac{dW_2}{d\theta} \qquad p_\phi = \frac{dW_3}{d\phi} \tag{7.83}$$

Substituting the value of p_ϕ from Eq. (7.80)

$$J_\phi = \oint p_\phi \, d\phi = \oint \alpha_\phi d\phi = 2\pi\alpha_\phi = 2\pi p_\phi \tag{7.84}$$

Substituting the value of p_θ, using Eqs. (7.83) and (7.81)

$$J_\theta = \oint p_\theta d\theta = \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}} \, d\theta \tag{7.85}$$

$$J_r = \oint p_r dr = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{\alpha_\theta^2}{r^2}} \, dr \tag{7.86}$$

Evaluation of the integral in Eq. (7.85) is done in a simple way. We have

$$H = \sum_i p_i \dot{q}_i - L$$

$$2T = \sum_i p_i \dot{q}_i = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} \tag{7.87}$$

Taking the plane in which the planet is moving as the plane for a plane polar coordinate system

$$2T = p_r \dot{r} + p_\psi \dot{\psi} \tag{7.88}$$

Equating the two expressions for $2T$

$$p_\theta \dot{\theta} = p_\psi \dot{\psi} - p_\phi \dot{\phi} \tag{7.89}$$

$$p_\theta d\theta = p_\psi d\psi - p_\phi d\phi$$

Consequently,

$$J_\theta = \oint p_\theta d\theta = \oint p_\psi d\psi - \oint p_\phi d\phi$$

As θ goes through a complete cycle, ϕ and ψ vary by 2π. Hence,

$$J_\theta = 2\pi p_\psi - 2\pi p_\phi \tag{7.90}$$

The Hamiltonian of the system in (r, ψ) co-ordinates is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2} p_\psi^2 \right) \tag{7.91}$$

Comparing the two expressions for the Hamiltonians, Eq. (7.76) and Eq. (7.91)

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = p_\psi^2 \tag{7.92}$$

The left hand side of Eq. (7.92) is α_θ^2 and therefore

$$p_\psi^2 = \alpha_\theta^2 \tag{7.93}$$

Substituting this value of p_ψ in Eq. (7.90)

$$J_\theta = 2\pi\left(\alpha_\theta - \alpha_\phi\right) \tag{7.94}$$

$$\alpha_\theta = \frac{J_\theta}{2\pi} + \alpha_\phi = \frac{J_\theta}{2\pi} + \frac{J_\phi}{2\pi} \tag{7.95}$$

With this value of α_θ, Eq. (7.86) takes the form

$$J_r = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{\left(J_\theta + J_\phi\right)^2}{4\pi^2 r^2}}\, dr \tag{7.96}$$

This integral can be evaluated by complex integration. Its value is given by

$$J_r = -(J_\theta + J_\phi) + \pi k \sqrt{\frac{2m}{-E}}$$

Solving for E

$$H = E = -\frac{2\pi^2 k^2 m}{(J_r + J_\theta + J_\phi)^2} \tag{7.97}$$

As expected, with an inverse square law force, energy is negative for bound orbits. Using Eq. (7.61), we have

$$\nu_r = \nu_\theta = \nu_\phi = \nu = \frac{4\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^3} \tag{7.98}$$

As all frequencies are equal, the motion is completely degenerate. From Eq. (7.97)

$$J_r + J_\theta + J_\phi = \left(\frac{2\pi^2 mk^2}{-E}\right)^{1/2} \tag{7.99}$$

With this value of $J_r + J_\theta + J_\phi$

$$\tau = \frac{1}{\nu} = \frac{(J_r + J_\theta + J_\phi)^3}{4\pi^2 mk^2} = \pi k \left(\frac{m}{-2E^3}\right)^{1/2} \tag{7.100}$$

From Eq. (5.59), the semi-major axis

$$a = \frac{-k}{2E} \tag{7.101}$$

Combining Eqs. (7.100) and (7.101)

$$\tau^2 = \frac{4\pi^2 m}{k} a^3 \tag{7.102}$$

which is Kepler's third law. Thus, the action-angle variables method gives frequencies of periodic motion without solving the complete equations of motion.

With a closed orbit, the motion is periodic and therefore completely degenerate. The degenerate frequencies can be eliminated by a canonical transformation to a new set of action-angle variables. With the new variables, two of the frequencies will be zero and the third one will be finite. In terms of the new variables, the Hamiltonian is

$$H = \frac{-2\pi^2 mk^2}{J_3^2} \qquad J_3 = J_r + J_\theta + J_\phi \tag{7.103}$$

which is the one corresponding to the finite frequency.

The action-angle variables are of use in fixing the location of planetary orbits and to determine the size and shape of orbits in space. They are also used to study the effects of perturbing forces in addition to the force between the two bodies.

7.9 ROAD TO QUANTIZATION

The action-angle variables of H-J theory played a very crucial role in the transition from classical to quantum theory. According to classical mechanics, these variables possess a continuous range of values.

Towards the end of the 19th century, experimental observations like blackbody radiation curves, photoelectric effect, *etc*. and discoveries such as electron, X-rays and radioactivity were carried out. These were very different things from anything that had appeared before. To explain blackbody radiation curves, Planck quantized energy and Bohr quantized angular momentum in his theory of the hydrogen atom. In 1915, Wilson and Sommerfeld proposed a general quantization rule which was applicable to all periodic systems. According to them, stationary states are those for which the proper action integral of any periodic motion equals an integer times *h*, the Planck's constant:

$$J = \oint p_i dq_i = n_i h \qquad n_i = 1, 2, 3, \ldots \qquad (7.104)$$

where q_1, q_2,..., q_n and p_1, p_2,..., p_n are the generalized coordinates and generalized momenta of the system. Proper action variables mean J 's whose frequencies are non-degenerate and different from zero.

In circular orbits, the momentum conjugate to the generalised coordinate *f* is the angular momentum. Hence, for circular orbits, Eq. (7.104) becomes

$$\oint mv\, r d\phi = nh \qquad \text{or} \qquad mvr \oint d\phi = nh$$

$$mvr = \frac{nh}{2\pi} \qquad n = 1,\ 2, 3, \ldots \qquad (7.105)$$

which is Bohr's frequency condition. The general quantization rule, Eq. (7.104), worked very well with all periodic systems in old quantum theory.

In the words of Sommerfeld, the method of action-angle variables is the bridge between classical physics and quantum theory. Thus, the Hamilton-Jacobi theory in action-angle variables provided a royal road to quantization.

WORKED EXAMPLES

Example 7.1 Apply the Hamilton-Jacobi method to study the motion of a freely falling body.

Solution: Let the vertical direction be the z-axis and the zero of the potential energy be the ground level. The system has only one degree of freedom and the coordinate is the value of z. The Hamiltonian

$$H = \frac{p^2}{2m} + mgz = E \qquad p = \frac{\partial S}{\partial z} \tag{i}$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + mgz + \frac{\partial S}{\partial t} = 0 \tag{ii}$$

Let the solution of the equation is of the form

$$S(z, \alpha, t) = W(z, \alpha) - \alpha t \tag{iii}$$

Since $\dfrac{\partial S}{\partial z} = \dfrac{\partial W}{\partial z}$ and $\dfrac{\partial S}{\partial t} = -\alpha$, Eq. (ii) takes the form

$$\frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz - \alpha = 0 \tag{iv}$$

$$\frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz)} \qquad \text{or} \qquad W = \int \sqrt{2m(\alpha - mgz)}dz + C \tag{v}$$

where C is the integration constant. Consequently,

$$S = \int \sqrt{2m(\alpha - mgz)}\, dz - \alpha t + C \tag{vi}$$

$$\frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{2} \int \frac{dz}{\sqrt{\alpha - mgz}} - t = \beta$$

$$\beta + t = \frac{\sqrt{2m}}{2}\,\frac{2\sqrt{(\alpha - mgz)}}{-mg} = -\sqrt{\frac{2}{m}}\,\frac{\sqrt{\alpha - mgz}}{g}$$

Squaring and rearranging

$$z = \frac{\alpha}{mg} - \frac{(\beta + t)^2 g}{2} \tag{vii}$$

When $t = 0$, $z = z_0$ and $p = 0$. From Eq. (v) we have

$$0 = p = \frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz_0)} \qquad \text{or} \qquad \alpha = mgz_0 \tag{viii}$$

Substituting this value of α and the initial conditions in Eq. (vii)

$$z_0 = \frac{mgz_0}{mg} - \frac{\beta^2 g}{2} \qquad \text{or} \qquad \beta = 0$$

Consequently from Eq. (vii), the equation of the freely falling body is

$$z = z_0 - \frac{1}{2} g t^2 \tag{ix}$$

Example 7.2 Use action-angle variables to obtain the energy levels of the hydrogen atom.

Solution: In the hydrogen atom the electron revolves around the nucleus in circular orbits. The potential energy $V(r) = \frac{-e^2}{4\pi\varepsilon_0 r} = -\frac{k}{r}$ $k = \frac{e^2}{4\pi\varepsilon_0}$

In spherical polar coordinates

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{e^2}{4\pi\varepsilon_0 r} \qquad \text{(i)}$$

which is the same as the Hamiltonian of the Kepler problem, Eq. (7.76), with $k = e^2/4\pi\varepsilon_0$ where ε_0 is the permittivity of vacuum. The energy of the system, given in Eq. (7.103), is

$$E = \frac{-2\pi^2 k^2 m}{J_3^2} \qquad \text{(ii)}$$

By Wilson-Sommerfeld quantization rule, (Eq. [7.104])

$$J_3 = nh \qquad n = 1, 2, 3, \ldots,$$

Substituting the values of J_3 and k in Eq. (ii)

$$E = -\frac{me^4}{8\varepsilon_0^2 n^2 h^2} = -\frac{me^4}{32\pi^2\varepsilon_0^2 n^2 \hbar^2} \qquad \text{(iii)}$$

where $\hbar = h/2\pi$ is the modified Planck's constant.

Example 7.3 Consider the motion of a particle of mass *m* and charge *q* circling around a uniform magnetic field ***B***, along the *z*-axis, generated by a static vector potential $\mathbf{A} = (\mathbf{B} \times \mathbf{r})/2$. Show that the magnetic moment *m* of the particle is an invariant quantity.

Solution: Since the magnetic field is along the *z*-axis, $B_x = 0$, $B_y = 0$, $B_z = B$. The static vector potential

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = -\hat{i}\frac{By}{2} + \hat{j}\frac{Bx}{2}$$

Let us consider the motion of the particle in cylindrical co-ordinates. Potential

$$V = -\frac{q}{c}\mathbf{A} \cdot \mathbf{v} = -\frac{q}{c}(A_x v_x + A_y v_y + A_z v_z)$$

Since $A_z = 0$

$$V = -\frac{q}{c}\left(-\frac{Byv_x}{2} + \frac{Bxv_y}{2}\right) = -\frac{qB}{2c}(x\dot{y} - y\dot{x})$$

In cylindrical co-ordinates

$$x = \rho\cos\phi \qquad \dot{x} = -\rho\sin\phi\dot{\phi}$$

$$y = \rho\sin\phi \qquad \dot{y} = \rho\cos\phi\dot{\phi}$$

Substituting the values of x, y, $\dot{x}$ and $\dot{y}$

$$V = -\frac{qB}{2c}(\rho^2 \cos^2\phi\dot{\phi} + \rho^2 \sin^2\phi\dot{\phi})$$

$$= -\frac{qB}{2c}\rho^2\dot{\phi}$$

Lagrangian $$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{qB}{2c}\rho^2\dot{\phi}$$

Since ϕ is a cyclic co-ordinate, p_ϕ is a constant of motion

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} + \frac{qB}{2c}\rho^2$$

The circulating frequency is the same as the cyclotron frequency (ω_c):

$$\omega_c = \dot{\phi} = \frac{qB}{mc}$$

Substituting this value of $\dot{\phi}$ in the expression for p_ϕ

$$p_\phi = \frac{3}{2}\frac{qB}{c}\rho^2$$

$$J_\phi = \oint p_\phi\, d\phi = \frac{3}{2}\frac{qB}{c}\rho^2 \times 2\pi = \frac{3\pi qB\rho^2}{c}$$

Magnetic moment $\mu = \frac{q}{2m}\mathbf{L}$ where the angular momentum $\mathbf{L} = m\mathbf{v}_\perp\rho$. Since

$$\mathbf{v}_\perp = \rho\dot{\phi} = \rho\frac{qB}{mc}$$

$$\mu = \frac{q}{2m}m\frac{\rho qB}{mc}\rho = \frac{q^2\rho^2 B}{2mc}$$

$$= \frac{q^2 B}{2mc}\frac{J_\phi c}{3qB\pi} = \frac{qJ_\phi}{6\pi m} = \text{constant}$$

Example 7.4 Consider a particle of mass *m* moving in a potential $V(r)$. Write the Hamilton-Jacobi equation in spherical polar coordinates and show that

$$p_\theta^2 + \left(\frac{p_\phi^2}{\sin^2\theta}\right)$$

is a constant, where p_q and p_f are the conjugate momenta corresponding to the coordinates q and f.

Solution: From Eqs. (7.80) and (7.81), we have

$$\frac{dW_3}{d\phi} = \alpha_\phi = \text{constant}$$

$$\left(\frac{dW_2}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 = \text{constant}$$

Since $$\frac{dW_2}{d\theta} = p_\theta \quad \text{and} \quad \frac{dW_3}{d\phi} = p_\phi$$

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = \alpha_\theta^2 = \text{constant.}$$

Example 7.5 In the inverse square force field, elliptic orbits are described by

$$\frac{1}{r} = \frac{mk}{L^2}(1 + \epsilon\cos\theta) \qquad \epsilon^2 = 1 + \frac{2EL^2}{mk^2}$$

where (r, θ) are the polar co-ordinates of the object of mass m, L is the orbital angular momentum and k is a constant. Obtain an expression for energy in terms of phase integrals. Given that

$$\oint \frac{\epsilon^2 \sin^2\theta\, d\theta}{(1 + \epsilon\cos\theta)^2} = 2\pi\left(\frac{1}{\sqrt{1 - \epsilon^2}} - 1\right)$$

Solution: The phase integrals of the system are

$$J_r = \oint p_r \, dr \qquad \text{and} \qquad J_\theta = \oint p_\theta \, d\theta$$

where, $\quad p_r = m\dot{r} \qquad \text{and} \qquad p_\theta = L = mr^2\dot{\theta} = \text{constant}$

The second integral gives

$$J_\theta = p_\theta \oint d\theta = 2\pi p_\theta \quad \text{or} \quad p_\theta = \frac{J_\theta}{2\pi}$$

$$p_r \, dr = m \frac{dr}{dt} dr = m \frac{dr}{d\theta} \frac{d\theta}{dt} dr = m \frac{dr}{d\theta} \dot{\theta} \frac{dr}{d\theta} d\theta$$

$$= mr^2\dot{\theta} \left(\frac{1}{r} \frac{dr}{d\theta} \right)^2 d\theta = p_\theta \left(\frac{1}{r} \frac{dr}{d\theta} \right)^2 d\theta$$

We have

$$\frac{1}{r} = \frac{mk}{L^2} (1 + \in \cos\theta) \qquad L = mr^2\dot{\theta}$$

Differentiating with respect to θ

$$-\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{mk}{L^2}\in\sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = r\frac{mk}{L^2}\in\sin\theta = \frac{\in\sin\theta}{1+\in\cos\theta}$$

$$J_r = \oint p_r\,dr = \oint p_\theta\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 d\theta = p_\theta\oint\frac{\in^2\sin^2\theta\,d\theta}{(1+\in\cos\theta)^2}$$

$$= p_\theta\,2\pi\left[\frac{1}{\sqrt{1-\in^2}}-1\right] = J_\theta\left[\frac{1}{\sqrt{1-\in^2}}-1\right]$$

$$= \frac{J_\theta}{\sqrt{1-\in^2}} - J_\theta$$

$$J_r + J_\theta = \frac{J_\theta}{\sqrt{1-\in^2}} \qquad \text{or} \qquad J_r + J_\theta = \frac{J_\theta}{\sqrt{-2EL^2/mk^2}}$$

$$(J_r + J_\theta)^2 = -\frac{J_\theta^2}{2EL^2/mk^2}$$

Since $\qquad p_\theta = J_\theta/2\pi \qquad$ and $\qquad p_\theta = L$

$$J_\theta^2 = 4\pi^2 L^2$$

Hence,

$$(J_r + J_\theta)^2 = -\frac{2\pi^2 mk^2}{E}$$

$$E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta)^2}$$

REVIEW QUESTIONS

1. Outline the Hamilton-Jacobi theory.
2. State and explain the Hamilton-Jacobi equation for Hamilton’s principal function.
3. Show that Hamilton’s principal function is a generator of a canonical transformation to constant coordinates and momenta.

4. Explain the physical significance of Hamilton's principal function.
5. Explain the Hamilton-Jacobi equation for Hamilton's characteristic function.
6. What are action and angle variables?
7. Outline how action-angle variables can be used to obtain the frequencies of a periodic system.
8. Explain how the method of action-angle variables provides a procedure for quantization of systems.
9. State and explain the Wilson-Sommerfeld quantization rule.

PROBLEMS

1. For a harmonic oscillator, show that Hamilton's principal function is equal to the time integral of the Lagrangian.
2. Consider a particle of mass m moving in a potential $V(r)$. Write the Hamilton-Jacobi equation in spherical polar coordinates and reduce them to quadratures.
3. Solve the problem of projectile of mass m in a vertical plane by using the Hamilton-Jacobi method. Find the equation of the trajectory and the dependence of coordinates on time. Assume that the projectile is fired off at time $t = 0$ from the origin with a velocity $\mathbf{v}_0$, making an angle with the horizontal.
4. Deduce the Hamilton-Jacobi equation of a spinning top. Separate the variables in it and reduce it to quadratures.
5. Show that Bohr's quantization rule is a consequence of the quantization rule of Wilson and Sommerfeld.

8

The Motion of Rigid Bodies

We have been considering the motion of particles in the earlier chapters. In this chapter we consider motion of rigid bodies, which require six generalized co-ordinates for the specification of their configurations. A general displacement of a rigid body can be considered to be a combination of translations and rotations. Its motion can be described in two co-ordinate systems, one an inertial system which is fixed in space and the other a body system fixed to the rigid body. Of the six independent co-ordinates, three are used to specify the rotational motion. We now discuss the characteristics and the dynamical equations of motion of some important systems of rigid body motion.

8.1 INTRODUCTION

A body is said to be *rigid* if the relative position of parts of the body remains unchanged during motion or under the action of external forces. During motion the body as a whole moves. It can also be considered a system consisting of a large number of particles such that the distances between pairs of particles remain constant. That is, $r_{ij} = c_{ij}$, where r_{ij} is the distance between the ith and jth particles, and c_{ij}'s are constants.

Next, we try to find the number of independent co-ordinates needed to describe the position of a rigid body in space. A rigid body in space is defined by three points which do not lie on the same straight line. Each point is specified by three co-ordinates and therefore 9 co-ordinates are needed to specify a rigid body. But these 9 co-ordinates are connected by the 3 equations of constraints $r_{12} = c_{12}$ $r_{13} = c_{13}$ and $r_{23} = c_{23}$

Hence we require 6 co-ordinates to specify the position of a rigid body. Apart from the constraints of rigidity, there may be additional constraints on the rigid body; for example, the body may be constrained to move on a surface, or it may be allowed to move with one point fixed. These additional constraints will further reduce the number of independent co-ordinates. There are several ways of selecting the independent co-ordinates.

The two important types of motion of a rigid body are *translational motion* and *rotational motion*. The translation of a rigid body will be given by the translation of any point in it, say the centre of mass, which behaves like a single particle in motion. The remaining three co-ordinates are used to specify the rotational motion.

8.2 ANGULAR MOMENTUM

Consider a rigid body consisting of n particles of mass m_i, i = 1, 2, 3,..., n. Let the body rotate with an instantaneous angular velocity w about an axis passing through the centre of mass, say O, and the co-ordinate system Oxyz is fixed in the body with its origin at O (see Fig. 8.1). Let the radius vector of the particle of mass m_i be $\mathbf{r}_i$. Then its instantaneous translational velocity $\mathbf{v}_i$ is given by

$$\mathbf{v}_i = \boldsymbol{\omega} \times r_i \qquad (8.1)$$

where w is the angular velocity of the body whose components are w_x, w_y and w_z.

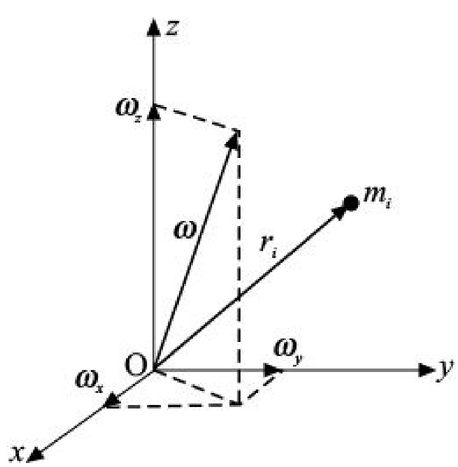


Fig. 8.1 Rigid body rotating with angular velocity w about an axis passing through the fixed point O.

The angular momentum about the origin O is

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \, \mathbf{r}_i \times \mathbf{v}_i \tag{8.2}$$

$$= \sum_i m_i \mathbf{r}_{\mathrm{i}} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{i}}) \tag{8.2a}$$

Using the vector triple product identity

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \tag{8.3}$$

$$\mathbf{L} = \sum_i m_i [r_i^2 \boldsymbol{\omega} - \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega})] \tag{8.4}$$

$$= \sum_i m_i[(x_i^2 + y_i^2 + z_i^2)(\hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z)$$
$$- (\hat{i}x_i + \hat{j}y_i + \hat{k}z_i)(x_i\omega_x + y_i\omega_y + z_i\omega_z)]$$
$$= \sum_i m_i\hat{i}(y_i^2\omega_x + z_i^2\omega_x - x_i y_i\omega_y - x_i z_i\omega_z)$$
$$+ \sum_i m_i\hat{j}(x_i^2\omega_y + z_i^2\omega_y - y_i x_i\omega_x - y_i z_i\omega_z)$$
$$+ \sum_i m_i\hat{k}(x_i^2\omega_z + y_i^2\omega_z - z_i x_i\omega_x - z_i y_i\omega_y) \quad (8.5)$$

In Eq. (8.5)

$$\sum_i m_i(y_i^2 + z_i^2) = \sum_i m_i(r_i^2 - x_i^2) \qquad \sum_i m_i(x_i^2 + z_i^2) = \sum_i m_i(r_i^2 - y_i^2)$$
$$\sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i(r_i^2 - z_i^2)$$

Let us now define the following 9 quantities:

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2) \qquad I_{xy} = I_{yx} = -\sum_i m_i x_i y_i \quad (8.6a)$$
$$I_{yy} = \sum_i m_i(r_i^2 - y_i^2) \qquad I_{yz} = I_{zy} = -\sum_i m_i y_i z_i \quad (8.6b)$$
$$I_{zz} = \sum_i m_i(r_i^2 - z_i^2) \qquad I_{zx} = I_{xz} = -\sum_i m_i z_i x_i \quad (8.6c)$$

In terms of these quantities, Eq. (8.5) can be written as

$$\mathbf{L} = \hat{i}(\omega_x I_{xx} + \omega_y I_{xy} + \omega_z I_{xz})$$
$$+ \hat{j}(\omega_x I_{yx} + \omega_y I_{yy} + \omega_z I_{yz})$$
$$+ \hat{k}(\omega_x I_{zx} + \omega_y I_{zy} + \omega_z I_{zz}) \quad (8.7)$$

where the coefficients I_{xx}, I_{yy} and I_{zz} involve the sums of the squares of the co-ordinates and are referred to as **moments of inertia** of the body about the co-ordinate axes: I_{xx}–moment of inertia about the *x*-axis, I_{yy}–moment of inertia about the *y*-axis and I_{zz}–moment of interia about the *z*-axis. The coefficients I_{xy}, I_{yz},... involve the sums of the products of the co-ordinates and are called the

products of inertia. The components of **L** may be written in a compact form as

$$L_\alpha = \sum_\beta \omega_\beta \, I_{\alpha\beta} \qquad \alpha = x, y, z \qquad \beta = x, y, z \qquad (8.8)$$

In the above, the body rotates about a general direction. If the body is rotating about the z-axis, *w* =(0, 0, *w*) from Eq.(8.7). We now have

$$L_x = \omega I_{xz} \qquad L_y = \omega I_{yz} \qquad L_z = \omega I_{zz} \qquad (8.9)$$

That is, the angular momentum vector has components in all the three directions, indicating that **L** and w are not in the same direction. This leads us to the important result that **L** is not necessarily always in the same direction as the instantaneous axis of rotation.

8.3 KINETIC ENERGY

We now derive a general expression for the rotational kinetic energy of a rigid body. Consider a rigid body rotating about an axis passing through a fixed point in it with an angular velocity w. A particle of mass m_i at a distance $\mathbf{r}_i$ has a velocity $\mathbf{v}_i$ given by Eq. (8.1). The kinetic energy of the whole body is given by

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \mathbf{v}_i^2 = \frac{1}{2} \sum_i (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (m_i \mathbf{v}_i) \tag{8.10}$$

Using the result

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \tag{8.11}$$

$$T = \frac{1}{2} \sum_i \boldsymbol{\omega}.(\mathbf{r}_i \times m_i \mathbf{v}_i) \tag{8.12}$$

Since $\boldsymbol{\omega}$ is the same for all particles, using Eq. (8.2)

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \sum_i (\mathbf{r}_i \times m_i \mathbf{v}_i)$$

$$= \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} \tag{8.13}$$

Using Eq. (8.7) to replace **L**

$$T = \frac{1}{2}(\omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz}) + \omega_x \omega_y I_{xy} + \omega_y \omega_z I_{yz} + \omega_z \omega_x I_{zx} \tag{8.14}$$

In a more compact form

$$T = \frac{1}{2} \sum_{\beta} \omega_\alpha \omega_\beta I_{\alpha\beta} \tag{8.15}$$

It may be noted from Eqs. (8.7) and (8.14) that

$$L_x = \frac{\partial T}{\partial \omega_x} \qquad L_y = \frac{\partial T}{\partial \omega_y} \qquad L_z = \frac{\partial T}{\partial \omega_z} \tag{8.16}$$

Defining a new vector r by

$$\boldsymbol{\rho} = \frac{\boldsymbol{\omega}}{\sqrt{2T}} \tag{8.17}$$

Equation (8.14) reduces to

$$I_{xx}\rho_x^2 + I_{yy}\rho_y^2 + I_{zz}\rho_z^2 + 2I_{xy}\rho_x\rho_y + 2I_{yz}\rho_y\rho_z + 2I_{zx}\rho_z\rho_x = 1 \tag{8.18}$$

The ellipsoid described by Eq. (8.18) is called the **Poinsots ellipsoid of inertia**.

8.4 INERTIA TENSOR

We considered a rigid body as one consisting of discrete, separate particles. In reality, the situation is different and the rigid body is continuous. Again, the density *r* may not be constant over the entire body. Hence, it is more appropriate to replace summation by integration in the above equations. In that case the moments of inertia and products of inertia take the form

$$I_{xx} = \int_V \rho(r^2 - x^2)\, dV \qquad I_{xy} = I_{yx} = -\int_V \rho\, xy\, dV$$

$$I_{yy} = \int_V \rho(r^2 - y^2)\, dV \qquad I_{yz} = I_{zy} = -\int_V \rho\, yz\, dV \qquad (8.19)$$

$$I_{zz} = \int_V \rho(r^2 - z^2)\, dV \qquad I_{zx} = I_{xz} = -\int_V \rho\, zx\, dV$$

With a slight change in notation, all the 9 coefficients in Eq. (8.19) can be combined into a single one. As the co-ordinate axes can be denoted by x_j, j = 1, 2, 3 the coefficient I_{jk} (k = 1, 2, 3) can be written as

$$I_{jk} = \int_V \rho(r)\, (r^2\delta_{jk} - x_j x_k)\, dV \qquad (8.20)$$

where d_{jk} is the Kronecker *d* -symbol. It is obvious from the expression for angular momentum, Eq. (8.7), that the components of *L* are linear functions of *w* which we may write in matrix notation as

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad (8.21)$$

The 9 elements I_1 , I_2 , ... of the 3 3 matrix may be considered the components of a single entity **I**, called a **tensor**. As the products of inertia satisfy the symmetry relation $I_{12} = I_{21}$, $I_{23} = I_{32}$, $I_{13} = I_{31}$, the tensor **I** is a **symmetric tensor**. Now the relations connecting the components of **L** and w, Eq. (8.7), can be written as

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} \tag{8.22}$$

where the quantity **I** is a second rank tensor and is usually called the **moment of inertia tensor** or briefly the **inertia tensor.** From Eq. (8.22), we may conclude that the product of a tensor and a vector is a vector. In dyadic notation, it is written as

$$\mathbf{I} = m_i(r_i^2\mathbf{1} - \mathbf{r}_i\mathbf{r}_i) \tag{8.23}$$

where **1** in the first term is a unit dyadic. Then

$$\mathbf{I} \cdot \boldsymbol{\omega} = m_i[r_i^2\boldsymbol{\omega} - \mathbf{r}_i(\mathbf{r}_i . \boldsymbol{\omega})] = \mathbf{L} \tag{8.24}$$

in agreement with Eq. (8.4). In terms of inertia tensor, the kinetic energy can be written as

$$T = \frac{1}{2}\tilde{\boldsymbol{\omega}} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \tag{8.25}$$

where $\tilde{\boldsymbol{\omega}}$ is the transpose of the column matrix $\boldsymbol{\omega}$. In the expanded form, Eq. (8.25) is

$$T = \frac{1}{2}(\omega_1\ \omega_2\ \omega_3)\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{8.26}$$

$$T = \frac{1}{2}I\omega^2 \tag{8.27}$$

where I is a scalar, the moment of inertia about the axis of rotation. From Eq. (8.25) we conclude that the product of two vectors and a tensor is a scalar.

The value of moment of inertia depends upon the direction of the axis of rotation. If w changes its direction with respect to time, the moment of inertia must also be considered a function of time. The moment of inertia also depends upon the choice of origin of the body set of axes. Another important result, which can be easily proved, is Steiner’s theorem: *The moment of inertia about a given axis is equal to the moment of inertia about a parallel axis through the centre of mass plus the moment of inertia of the body, as if concentrated at the centre of mass, with respect to the original axis*.

8.5 PRINCIPAL AXES

The inertia tensor we defined is with respect to a co-ordinate system which is fixed to a point in the body. We can simplify the mathematical calculations considerably if we choose the co-ordinate axes in such a way that the off-diagonal elements vanish. As the inertia tensor is symmetric, it is always possible to orient the axes so that the products of inertia terms vanish. The axes of this co-ordinate system are known as the **principal axes** of the body. The origin of the principal axes system is called the **principal point**. The three co-ordinate planes, each of which passes through two principal axes, are called **principal planes** at the origin. In this system, the inertia tensor is diagonal and the three elements of the inertia tensor are called **principal moments of inertia**. It is the practice to use a single subscript for the principal moments to distinguish them from moments of inertia about arbitrary axes. The principal moments of inertia are thus denoted by I_1, I_2 and I_3. Since the principal axes are attached to the rigid body, I_1, I_2 and I_3 do not change with time. Therefore, they may be treated as constants. It is for this reason that moving axes attached to the body are employed. In the principal axes system

$$\mathbf{L} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3 \tag{8.28}$$

where $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are unit vectors along the three principal axes. The kinetic energy T takes the form

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 \tag{8.29}$$

Poinsot's ellipsoid reduces to

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1 \tag{8.30}$$

If the body is rotating about one of its principal axes, say the z-axis, in the principal axes system $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega$. Then,

$$L_1 = L_2 = 0 \text{ and } L_3 = I_3\omega$$

Consequently, the vector **L** has the same direction as $\boldsymbol{\omega}$. The same result is obtained for rotation of a body about other principal axes. In all the cases, we get

$$\mathbf{L} = I\,\boldsymbol{\omega} \tag{8.31}$$

where I is the corresponding principal moment of inertia.

Next, we shall see how to find the principal axes. Sometimes we may be able to fix up the principal axes by examining the symmetry of the body. In the general case, suppose we have the moments and products of inertia of a body with respect to an arbitrary set of *x*, *y* and *z*-axes. Then its angular momentum is given by Eq. (8.21). In addition, we require Eq. (8.31) to be true. This leads to

$$\begin{aligned} I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= I\omega_1 \\ I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 &= I\omega_2 \\ I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 &= I\omega_3 \end{aligned} \tag{8.32}$$

Rearranging,

$$\begin{aligned} (I_{11} - I)\,\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\,\omega_3 &= 0 \end{aligned} \tag{8.33}$$

For a solution to exist, the determinant of the coefficients must be zero. That is,

$$\begin{vmatrix} I_{11} - 1 & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0 \tag{8.34}$$

The expansion of the determinant leads to a cubic equation in *I*, whose 3 roots we shall denote by I_1, I_2 and I_3. These are the principal moments of inertia. The direction of the principal axis is found by substituting the corresponding *I* back into Eq. (8.33) and solving for w_1: w_2 : w_3. This ratio is just the ratio of the direction cosines relative to the original axes which specifies the direction of that principal axis.

Though, there are differences between a second rank tensor and a 3 3 matrix, we can make use of the properties of matrices in tensors. The elements of an inertia tensor **I** in a fixed co-ordinate system can be transformed into the elements of a tensor **I** in a rotating co-ordinate system by a **similarity transformation** $\mathbf{I}' = A\,\mathbf{I}\,\tilde{A}$ (8.35) where $\tilde{A}$ is the transpose of the orthogonal matrix *A*. This procedure is also equally good for obtaining the principal moments of inertia (see Example 8.3).

Rigid bodies are classified into three categories, depending on their principal moments of inertia: **Spherical top:** $I_1 = I_2 = I_3$. Any three mutually perpendicular axes can be selected as the principal axes.

Symmetric top: $I_1 = I_2 < I_3$ or $I_1 < I_2 = I_3$. Two principal moments of inertia are equal. Bodies belonging to the first type are called *oblate symmetric top* and those of the second type are called *prolate symmetric top*.

Asymmetric top: $I_1 \neq I_2 \neq I_3$.

8.6 EULER'S ANGLES

The angular momentum **L** and angular velocity w need not be parallel vectors as **I** is a tensor. The value of **I** with respect to a fixed frame, called the *space fixed frame* or *laboratory frame*, does not remain constant but changes as the body rotates. To have a constant value, it must be expressed in a frame, called the *body frame*, that is attached to the body.

To specify the position of a rigid body, 6 co-ordinates must be specified. Invariably, 3 of these are taken to be the co-ordinates of the centre of mass of the body. The other three co-ordinates are taken to be the angles that describe the orientation of the body axes with respect to the space-fixed axes. Though several choices are available, Euler's angles are the most commonly used ones.

Let the co-ordinate system that is fixed to the rigid body be O*xyz* and the one fixed to the space be $O'x'y'z'$. **Euler's angles** are the three successive angles of rotations involved when we go from the primed to the unprimed system. These are illustrated in Fig. 8.2. The transformation may be represented by the matrix equation $\mathbf{x} = R\mathbf{x}'$ (8.35a) **Step 1:** Rotate $x'y'z'$ axes anticlockwise through an angle *f* about the

z-axes. Let the resulting co-ordinate system be $x''y''z''$, z'' will be the same as z' (see Fig. 8.2a). The $x''y''$ plane will be the same as the $x'y'$ plane. The angle *f* is called the **precession angle.** The transformation matrix for this rotation is given by

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.36}$$

Step 2: Rotate $x''\ y''\ z''$ -system anticlockwise through an angle θ about the x'' axis. We get the system $x'''\ y'''\ z'''$ (see Fig. 8.2b). The angle θ is called the **nutation angle**. The transformation matrix for this rotation is

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{8.37}$$

Step 3: Rotate the $x'''\ y'''\ z'''$ axes again anticlockwise through an angle ψ about the z'''-axis which takes us to the desired O*xyz* axes. (see Fig. 8.2c.) The angle ψ is called the **body angle** and the transformation matrix for this rotation is

$$R_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.38}$$

Combining all the three rotations, the transformation from $x'y'z'$ axes to the body-fixed *xyz* axes can be written as

$$\mathbf{x} = R_\psi \mathbf{x}''' = R_\psi R_\theta \mathbf{x}'' = R_\psi R_\theta R_\phi \mathbf{x}' = R\mathbf{x}' \tag{8.39}$$

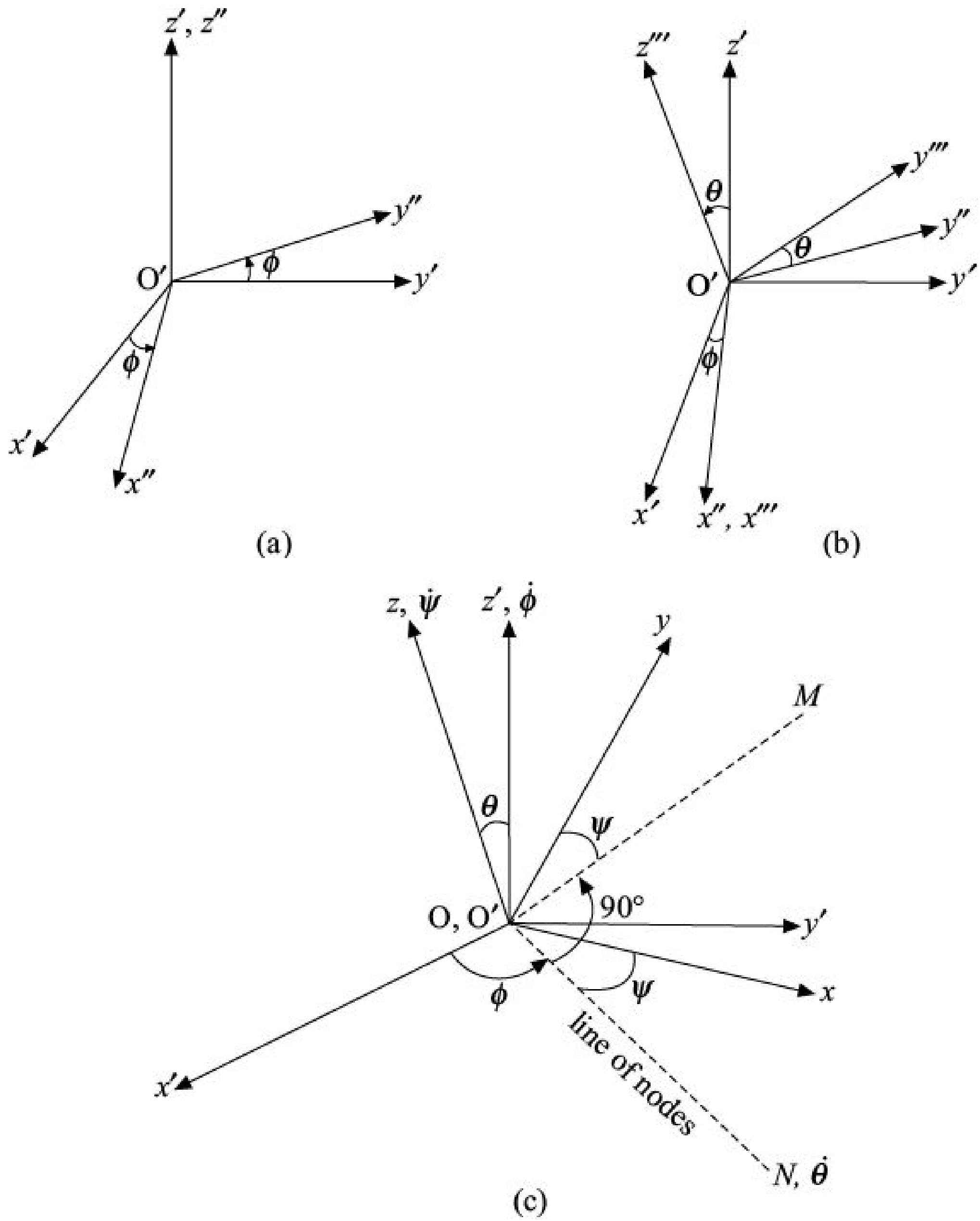


Fig. 8.2 Euler's angles (a) rotation through *f* (b) rotation through *q* (c) rotation through *y* and directions of angular velocities $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$.

where,

$R = R_\psi \ R_\theta \ R_\phi$ (8.40) A point to be kept in mind is that the angles *f*, *q* and *y* are measured in different planes. Substituting the values of R_y, R_q and R_f in Eq. (8.40), we get the matrix *R* as

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

(8.41)

The inverse transformation from the *xyz* axes to the *x y z* axes is **x** = R^{-1} **x** (8.42) R^{-1} is given by the transposed matrix $\tilde{R}$. The line *ON* formed by the intersection of *xy* and *x y* planes is called the **line of nodes** (see Fig. 8.2c). In Fig. 8.2c *q* is the angle between the axes *z* and *z*

y is the angle between *ON* and *Ox* measured in the *xy* plane *f* is the angle between *Ox*and *ON* measured in the *xy* plane.

The range of Euler's angles is $0 < \phi \le 2\pi \quad 0 \le \theta < \pi \quad 0 < \psi \le 2\pi$ (8.43) Angular velocity w is a vector pointing along the axis of rotation. The general infinitesimal rotation associated with vector w can be thought of as consisting of three successive infinitesimal rotations with angular velocities $\omega_\phi = \dot{\phi} \quad \omega_\theta = \dot{\theta} \quad \omega_\psi = \dot{\psi}$ (8.44) All infinitesimal rotations can be represented by vectors (see Section 8.7). This helps us to represent the three time derivatives $\omega_\phi = \dot{\phi}$ as detailed below:

$\omega_\phi = \dot{\phi}$ is directed along the z'-axis

$\omega_\theta = \dot{\theta}$ is directed along the line of nodes *ON* (8.45)

$\omega_\psi = \dot{\psi}$ is directed along the *z*-axis.

As rigid body rotations are described in terms of body-fixed co-ordinate systems, we must get the angular velocity vector $\boldsymbol{\omega}$ or its components ω_1, ω_2 and ω_3 in the body co-ordinate system. For this, we have to resolve $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ along the body axes. To simplify the analysis draw a line OM, located in the xy-plane and perpendicular to ON. Since ON is perpendicular to the zoz'-plane also, all lines perpendicular to ON will lie on the zoz' plane. Therefore, OM lies in the xy-plane as well as in the zoz'-plane. Hence, $\angle zOM = 90°$ and $\angle MOy = \psi$. The resolved components of $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ are listed below.

The component of

$$\begin{aligned} &\dot{\phi} \text{ along the } x\text{-axis} = \dot{\phi}\sin\theta\sin\psi \\ &\dot{\phi} \text{ along the } y\text{-axis} = \dot{\phi}\sin\theta\cos\psi \\ &\dot{\phi} \text{ along the } z\text{-axis} = \dot{\phi}\cos\theta \end{aligned} \tag{8.46}$$

The component of

$$\begin{aligned} &\dot{\theta} \text{ along the } x\text{-axis} = \dot{\theta}\cos\psi \\ &\dot{\theta} \text{ along the } y\text{-axis} = -\dot{\theta}\sin\psi \\ &\dot{\theta} \text{ along the } z\text{-axis} = 0 \end{aligned} \tag{8.47}$$

The component of

$$\begin{aligned} &\dot{\psi} \text{ along the } x\text{-axis} = 0 \\ &\dot{\psi} \text{ along the } y\text{-axis} = 0 \\ &\dot{\psi} \text{ along the } z\text{-axis} = \dot{\psi} \end{aligned} \tag{8.48}$$

Using these results, we can get the components of $\boldsymbol{\omega}$ along x-, y- and z-axes :

$$\begin{aligned} \omega_1 &= \omega_x = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \omega_y = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \omega_z = \dot{\phi}\cos\theta + \dot{\psi} \end{aligned} \tag{8.49}$$

These are called **Euler's geometrical equations**, which express the components of angular velocity with respect to the body axes in terms of Euler's angles and their time derivatives.

8.7 INFINITESIMAL ROTATIONS

While discussing Euler's angles, we associated vectors with infinitesimal rotations. We will now justify it by considering how vectors behave under rotation. Consider the change in the radius vector **r** of the point *M* produced by an infinitesimal anticlockwise rotation through an angle *df* about the axis of rotation. This is illustrated in Fig. 8.3.

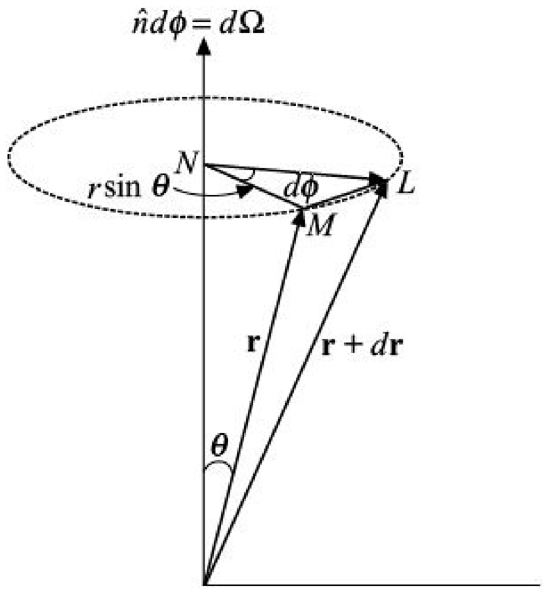


Fig. 8.3 An infinitesimal anticlockwise rotation of a vector.

The distance *ML* represents the magnitude of the change in the vector

$$ML = |dr| = r\sin\theta\, d\phi \qquad (8.50)$$

The direction of the vector *d***r** is along *ML* which is perpendicular to both **r** and $\hat{n}$ where $d\mathbf{W}$ is the unit vector along the axis of rotation. The direction of the vector *d***r** is the direction in which a right hand screw advances as vector $d\mathbf{r} = d\mathbf{W} \times \mathbf{r}$ is turned into vector **r**. Hence,

$$d\mathbf{r} = d\mathbf{\Omega} \times \mathbf{r} \tag{8.51}$$

From Eq. (8.51) we can state that the infinitesimal displacement $d\mathbf{r}$ is equivalent to an infinitesimal rotation $d\phi$ which can be represented by a vector $d\mathbf{\Omega} = \hat{n}d\phi$ pointing along the instantaneous axis of rotation. Dividing Eq. (8.51) by dt

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} \tag{8.52}$$

Generalizing, for a vector $\mathbf{A}$

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A} \tag{8.53}$$

8.8 RATE OF CHANGE OF A VECTOR

Rotational motion of rigid bodies is generally formulated in a body-fixed co-ordinate system. To convert results from a body-fixed system to a space fixed-system and vice versa, we should know how the time derivative of a vector in one system changes to the time derivative in the other system. Let O*xyz* be the co-ordinate system fixed to the rotating body and $x'y'z'$ be the space-fixed one with common origin. Let the unit vectors of the body fixed system be $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. The radius vector of a mass point at *P* of the body with respect to the body-fixed system is

$$\mathbf{r} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y + \hat{\mathbf{k}}z \tag{8.54}$$

As unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are constants with respect to the body-fixed system, specified by the subscript r

$$\left(\frac{d\mathbf{r}}{dt}\right)_r = \hat{\mathbf{i}}\frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} + \hat{\mathbf{k}}\frac{dz}{dt} \tag{8.55}$$

When we consider the time derivative of $\mathbf{r}$ with respect to the space-fixed system, the unit vectors also possess time derivatives as they change directions. Hence,

$$\left(\frac{d\mathbf{r}}{dt}\right)_s = \hat{\mathbf{i}}\ \ \frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} + \hat{\mathbf{k}}\frac{dz}{dt} + \frac{d\hat{\mathbf{i}}}{dt}x + \frac{d\hat{\mathbf{j}}}{dt}y + \frac{d\hat{\mathbf{k}}}{dt}z$$

$$= \left(\frac{d\mathbf{r}}{dt}\right)_r + \frac{d\hat{\mathbf{i}}}{dt}x + \frac{d\hat{\mathbf{j}}}{dt}y + \frac{d\hat{\mathbf{k}}}{dt}z \tag{8.56}$$

Using Eq. (8.52), we can write

$$\frac{d\hat{\mathbf{i}}}{dt}x = \boldsymbol{\omega} \times \hat{\mathbf{i}}x \qquad \frac{d\hat{\mathbf{j}}}{dt}y = \boldsymbol{\omega} \times \hat{\mathbf{j}}y \qquad \frac{d\hat{\mathbf{k}}}{dt}z = \boldsymbol{\omega} \times \hat{\mathbf{k}}z$$

Using Eq. (8.52), we can write

$$\left(\frac{d\mathbf{r}}{dt}\right)_s = \left(\frac{d\mathbf{r}}{dt}\right)_r + \boldsymbol{\omega} \times (\hat{\mathbf{i}}x + \hat{\mathbf{j}}y + \hat{\mathbf{k}}z)$$

$$= \left(\frac{d\mathrm{r}}{dt}\right)_r + (\boldsymbol{\omega} \times \mathbf{r}) \tag{8.57}$$

Generalizing to a general vector **A**

$$\left(\frac{d\mathbf{A}}{dt}\right)_s = \left(\frac{d\mathbf{A}}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{A} \tag{8.58}$$

From this, we get the important operator equation

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \boldsymbol{\omega} \times \tag{8.59}$$

where w is the angular velocity vector of the rotating body. Eq. (8.59) is a statement of the transformation of the time derivative between the body-fixed and space-fixed co-ordinate systems.

In Eq. (8.57), $(dr/dt)_S$ is the velocity $\mathbf{v}_S$ with respect to the space-fixed co-ordinate system and $(dr/dt)_r$ is the velocity $\mathbf{v}_r$ with respect to the rotating co-ordinate system. Eq. (8.57) can now be written as

$$\mathbf{v_s} = \mathbf{v_r} + \boldsymbol{\omega} \times \mathbf{r} \tag{8.60}$$

Equation (8.59) is the basic law on which the dynamical equation of motion of a rigid body is based.

8.9 CORIOLIS FORCE

Equation (8.59) can be used to obtain the relation connecting the inertial acceleration of the particle of mass *m* at *P* and its acceleration relative to the rotating frame. Using Eq. (8.59) to get the time rate of change of $\mathbf{v}_S$

$$\left(\frac{d\mathbf{v}_s}{dt}\right)_s = \left(\frac{d\mathbf{v}_s}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v}_s \tag{8.61}$$

Replacing $\mathbf{v}_s$ on the right side of Eq. (8.61) using Eq. (8.60), we get

$$\left(\frac{d\mathbf{v}_s}{dt}\right)_s = \left(\frac{d\mathbf{v}_r}{dt}\right)_r + \left(\frac{d(\boldsymbol{\omega} \times \mathbf{r})}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \left(\frac{d\mathbf{v}_r}{dt}\right)_r + \left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right)_r + \boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{8.62}$$

When angular velocity is constant, $(d\omega / dt) = 0$. The factor $(d\mathbf{v}_s / dt)_s$ is the inertial acceleration $\mathbf{a}_s$ of the particle relative to the inertial system, and $(d\mathbf{v}_r / dt)_r$ is the acceleration $\mathbf{a}_r$ of the particle relative to the rotating co-ordinate system. Then

$$\mathbf{a}_s = \mathbf{a}_r + 2(\boldsymbol{\omega} \times \mathbf{v}_r) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{8.63}$$

The equation of motion in the inertial system is

$$\mathbf{F} = m\mathbf{a}_s \tag{8.64}$$

Multiplying Eq. (8.63) by m and replacing $m\,\mathbf{a}_s$ by $\mathbf{F}$

$$\mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}_r) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{a}_r \tag{8.65}$$

To an observer in the rotating system, it appears as if the particle is moving under the influence of an effective force

$$\mathbf{F}_{eff} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}_r) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathrm{r}) \tag{8.66}$$

The third term on the right, $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, is called the **centrifugal force**. Its magnitude

$$|\, m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \,| = mr\omega^2 \sin\theta$$

where q is the angle between vectors w and **r**. This reduces to $-mr$w^2 when w is normal to the radius vector (circular motion). The negative sign indicates that the centrifugal force is directed away from the centre of rotation. It is not a real force, but a fictitious one. It is present only if we refer to moving co-ordinates in space.

The second term $2m(\mathbf{w} \times \mathbf{v}_r)$, called the **Coriolis force**, is present when a particle is moving in the rotating co-ordinate system. This is also not a real force, but a fictitious one. It is directly proportional to $\mathbf{v}_r$ and will disappear when there is no motion. Another feature of this force is that it does no work, since it acts in a direction perpendicular to velocity.

The centrifugal and coriolis forces are not due to any physical interaction, and hence they are non-inertial or fictitious forces. The rotating earth can be considered a rotating frame. Though its angular velocity is small, it has considerable effect on some of the quantities. Some of them are: (i) The Coriolis force has to be taken into account to compute accurately the trajectories of long range projectiles and missiles.

(ii) It is the Coriolis force on moving masses that produces a counterclockwise circulation in the northern hemisphere which affects the course of winds.

(iii) The spinning motion of the earth is that which causes the equatorial bulge.

8.10 EULER'S EQUATIONS OF MOTION

The inertia tensor **I** with respect to a body-fixed co-ordinate system is a constant. In a principal axes system also it is diagonal. Consider the rotation of a rigid body with one point fixed, which is taken as the origin of a body-fixed co-ordinate system. The rotational analogue of Newton's second law gives the rate of change of angular momentum with respect to a space-fixed co-ordinate. The torque **N** acting on the body

$$\mathbf{N} = \left(\frac{d\mathbf{L}}{dt}\right)_s \tag{8.67}$$

Equation (8.59) can be used to get the time derivative of **L** with respect to axes fixed in the body:

$$\left(\frac{d\mathbf{L}}{dt}\right)_s = \left(\frac{d\mathbf{L}}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{L} \tag{8.68}$$

Combining Eqs. (8.67) and (8.68)

$$\mathbf{N} = \left(\frac{d\mathbf{L}}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{L} \tag{8.69}$$

As the components of inertia tensor with respect to body axes are constants,

$$\mathbf{N} = \mathbf{I}\dot{\omega} + \boldsymbol{\omega} \times \mathbf{L} \tag{8.70}$$

If we select the principal axes of the body as body axes, from Eq. (8.70) we have

$$N_1 = I_1\dot{\omega}_1 + \omega_2 L_3 - \omega_3 L_2$$

But $\quad L_3 = I_3\omega_3,\ L_2 = I_2\omega_2$ and therefore

$$N_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)$$

Similar relations can be obtained for the other components also. Rearranging

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) + N_1 \\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) + N_2 \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) + N_3 \end{aligned} \tag{8.71}$$

Equations (8.71) are known as **Euler's equations** of motion for a rigid body with one point fixed . In the absence of external torques, Euler's equations reduce to simpler forms

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) \end{aligned} \tag{8.72}$$

The external torques acting on the earth are so weak that the rotational motion can be considered as torque-free in the first approximation.

The angular velocity and angular momentum are not parallel vectors. Though angular momentum is a conserved quantity, angular velocity is not. In general, the angular velocity will precess around the angular momentum vector and the angle between them varies in time. This is known as **nutation**.

8.11 FORCE-FREE MOTION OF A SYMMETRICAL TOP

As an example of Euler's equations of motion, we consider the special case in which the torque **N** = 0 and the body is a symmetrical top. A symmetrical top possesses an axis of symmetry and therefore two of the principal moments of inertia are equal. If the axis of symmetry is taken as the *z*-axis, $I_1 = I_2 = I \neq I_3$. For force-or torque-free motion, the centre of mass is either at rest or in uniform motion relative to the space-fixed inertial system. Therefore, we can take the centre of mass as the origin of the body-fixed co-ordinate system. In such a case the angular momentum arises only from rotation about the centre of mass. For such a symmetric body, Eq. (8.72) reduces to

$$I\dot{\omega}_1 = \omega_2\omega_3(I - I_3)$$
$$I\dot{\omega}_2 = \omega_3\omega_1(I_3 - I) \tag{8.73}$$
$$I_3\dot{\omega}_3 = 0$$

The last equation of Eq. (8.73) gives

$$\omega_3 = \text{constant in time} \tag{8.74}$$

Defining a constant k by

$$k = \frac{(I_3 - I)\omega_3}{I} \tag{8.75}$$

The first two equations of Eq. (8.73) become

$$\dot{\omega}_1 = -k\omega_2 \quad \text{and} \quad \dot{\omega}_2 = k\omega_1 \tag{8.76}$$

Differentiating the first one with respect to time and replacing $\dot{\omega}_2$ by $k\omega_1$

$$\ddot{\omega}_1 = -k\dot{\omega}_2 = -k^2\omega_1 \tag{8.77}$$

which is the equation for simple harmonic motion. Its solution is

$$\omega_1 = A\cos(kt + \theta_0) \tag{8.78}$$

where A and θ_0 are constants. From Eqs. (8.76) and (8.78)

$$\omega_2 = -\frac{1}{k}\dot{\omega}_1$$
$$\omega_2 = A\sin(kt + \theta_0) \tag{8.79}$$

Squaring Eqs. (8.78) and (8.79) and adding

Squaring Eqs. (8.78) and (8.79) and adding

$$\omega_1^2 + \omega_2^2 = A^2 = \text{constant} \tag{8.80}$$

Since ω_3 is a constant,

$$\omega = |\boldsymbol{\omega}| = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2} = (A^2 + \omega_3^2)^{1/2} = \text{constant} \tag{8.81}$$

Equations (8.78), (8.79) and (8.80) together suggest that the components ω_1 and ω_2 of ω trace out a circle of radius A with time in the xy-plane. Since the total angular velocity $|\boldsymbol{\omega}|$ is also a constant, this implies that the angular velocity vector $\boldsymbol{\omega}$ precesses in a cone about the z-axis (the body symmetry axis) with the angular frequency k. The frequency f_p and period T_p of this precession are given by

$$f_p = \frac{1}{T_p} = \frac{k}{2\pi} = \frac{(I_3 - I)\omega_3}{2\pi I} \tag{8.82}$$

The cone described by the angular velocity vector $\boldsymbol{\omega}$ is known as the **body cone**. In the body reference frame, its half angle α is given by

$$\tan\alpha = \frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_3} = \frac{A}{\omega_3} \tag{8.82a}$$

This precession is relative to the body-fixed axes which are themselves rotating in space with a larger frequency ω. Fig. 8.4 shows the precession of the angular velocity vector $\boldsymbol{\omega}$ about the body symmetry axis.

This precession is relative to the body-fixed axes which are themselves rotating in space with a larger frequency *w*. Fig. 8.4 shows the precession of the angular velocity vector w about the body symmetry axis.

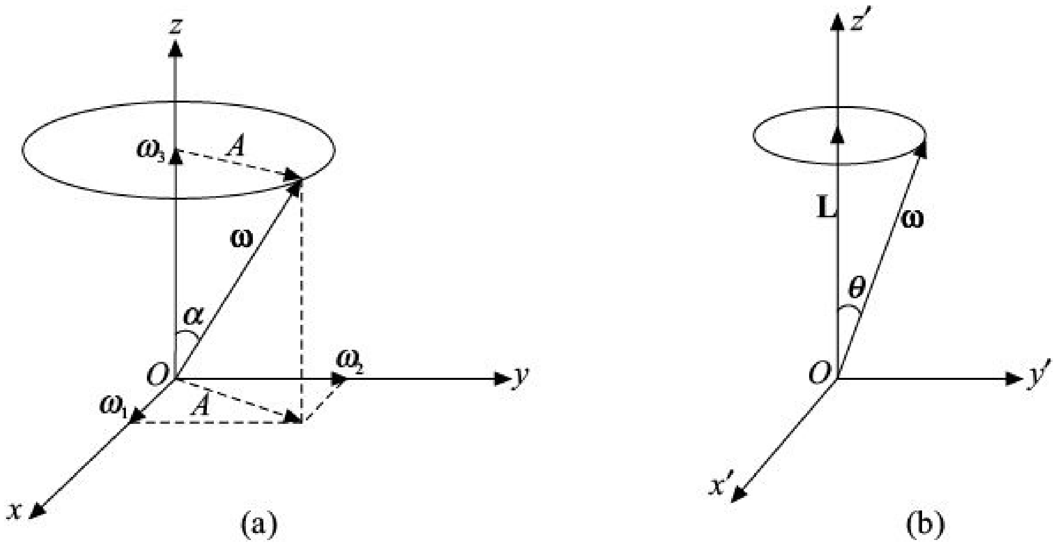


Fig. 8.4 (a) The precession of the angular velocity vector w about the body-fixed z-axis; (b) The angular velocity vector w precesses about the space-fixed z-axis.

The constants A and w_3 can be evaluated in term of the familiar constants, the kinetic energy T and angular momentum **L**.

$$T = \frac{1}{2}(I\omega_1^2 + I\omega_2^2 + I_3\omega_3^2) = \frac{1}{2}IA^2 + \frac{1}{2}I_3\omega_3^2 \qquad (8.83)$$

$$L^2 = I^2\omega_1^2 + I^2\omega_2^2 + I_3\omega_3^2 = I^2A^2 + I_3^2\omega_3^2 \qquad (8.84)$$

From Eqs. (8.83) and (8.84)

$$A^2 = \frac{L^2 - 2I_3T}{I(I - I_3)} \quad \text{and} \quad \omega_3^2 = \frac{L^2 - 2IT}{I_3(I_3 - I)} \qquad (8.85)$$

What we have been discussing so far is the precessional motion of w about an axis fixed in the body. As viewed from the space-fixed (inertial) system, there should be two constants of motion, the angular momentum **L** and kinetic energy *T*. In the body-fixed sytem, **L** acts in the direction of *z* as shown in Fig. 8.4(b) : **L** = constant (8.86) Since the centre of mass is fixed , the kinetic energy is completely rotational and is given by

$$T_{rot} = \frac{1}{2}\mathbf{L} \cdot \boldsymbol{\omega} = \text{constant} \qquad (8.87)$$

As **L**.w is constant, during motion w must move in such a way that its projection on the stationary **L** (the *z*-axis) is constant. That is, w must precess around and make a constant angle *q* with the angular momentum vector **L** (see Fig. 8.4b). From Eq. (8.87) the angle *q* between the vectors **L** and w is given by

$$\cos\theta = \frac{\mathbf{L} \cdot \boldsymbol{\omega}}{L\omega} = \frac{2T_{rot}}{L\omega} = \text{constant} \qquad (8.88)$$

This precession of w about the angular momentum vector **L** traces out a cone around **L**. This cone is known as the **space cone**. Thus, when viewed from the body co-ordinate system, the vector w precesses around the *z*-axis (symmetry axis) whereas it precesses around the *z*-axis when viewed in the space-fixed system. The situation may be described as shown in Fig. 8.5 (a), with the body cone rolling around the space cone with the line of contact along the direction of angular velocity w which precesses around the *z*-axis when viewed from the body-fixed frame and around the *z*-axis when viewed from the space-fixed frame. Depending on the value of I and I_3, the body cone may roll outside ($I_3 < I$) or inside ($I_3 > I$) the space cone as shown in Fig. 8.5.

An interesting example of the force-free motion of a symmetric body is provided by the rotation of the earth. The earth rotates freely about its polar axis. Its axis of rotation departs slightly from its symmetry axis. From astronomical measurements $(I_3 - I)/I$ is found to be 0.00329. From Eq. (8.82)

$$T_p = \frac{2\pi}{k} = \frac{2\pi I}{\omega_3 (I_3 - I)} = \frac{1 \text{ day}}{0.00329} \cong 304 \text{ days}$$

Recent measurements give T_p = 433 days. The discrepancy is probably due to the fact that the earth is not a perfectly rigid body. Thus, the earth's rotation axis precesses about the north pole in a circle of radius about 10 m with a period of about 433 days.

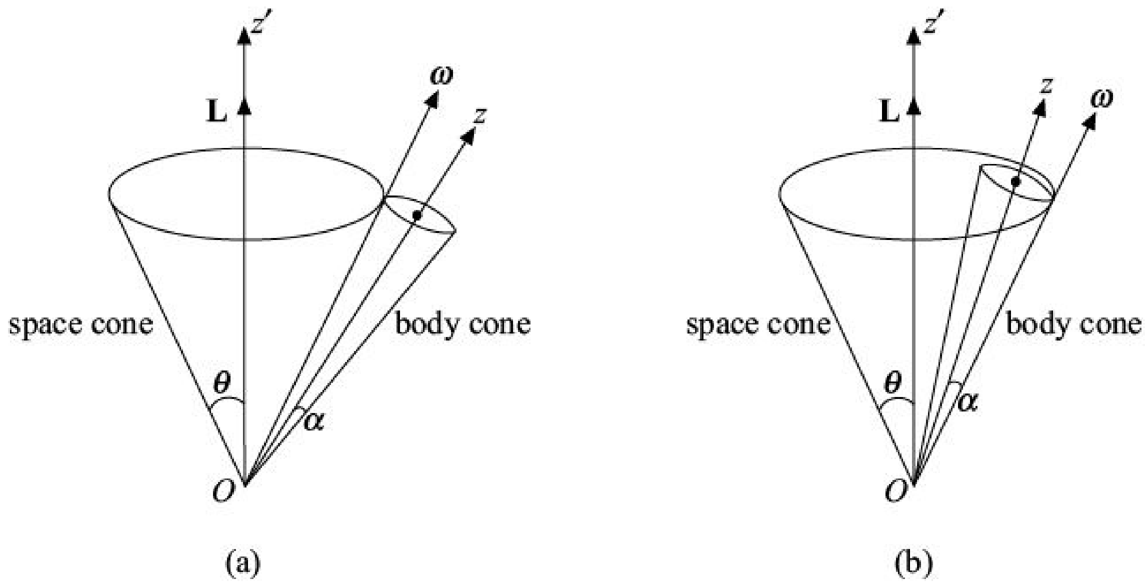


Fig. 8.5 (a) Space and body cones for $I_3 < I$; (b) Space and body cones for $I_3 > I$.

8.12 HEAVY SYMMETRIC TOP WITH ONE POINT FIXED

As a second example of rigid body dynamics, we consider the motion of a heavy symmetrical top spinning freely about its symmetry axis under the influence of a torque produced by its own weight. The symmetry axis of the body is one of its principal axes and we choose it as the *z*-axis of the co-ordinate system fixed in the body. The body is fixed at the point O which is on the symmetry axis but does not coincide with the centre of gravity. Point O is taken as the origin of the space-fixed $(x'y'z')$ and body-fixed (*xyz*) co-ordinate systems. Fig. 8.6 shows the spinning top along with the axes of the co-ordinate systems. We shall use the Euler's angles to describe the motion of the top. The line *ON* is the line of nodes; *q* gives the inclination of the *z*-axis from the *Oz* axis, *f* measures the azimuth of the symmetric top about the vertical, and ψ is the rotation angle of the top about its own *z*-axis. Let the distance of the centre of gravity *G* from O be *l*. As the symmetry axis is selected as the *z*- axis of the body-fixed system

$$I_1 = I_2 = I \neq I_3$$

As the translational kinetic energy is zero, the kinetic energy *T* is given by

$$T = \frac{1}{2}\left[I(\omega_1^2 + \omega_2^2) + I_3\omega_3^2 \right] \tag{8.89}$$

According to Euler's geometrical equations, Eq. (8.49)

$$\begin{aligned} \omega_1 &= \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi \\ \omega_2 &= -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned} \tag{8.90}$$

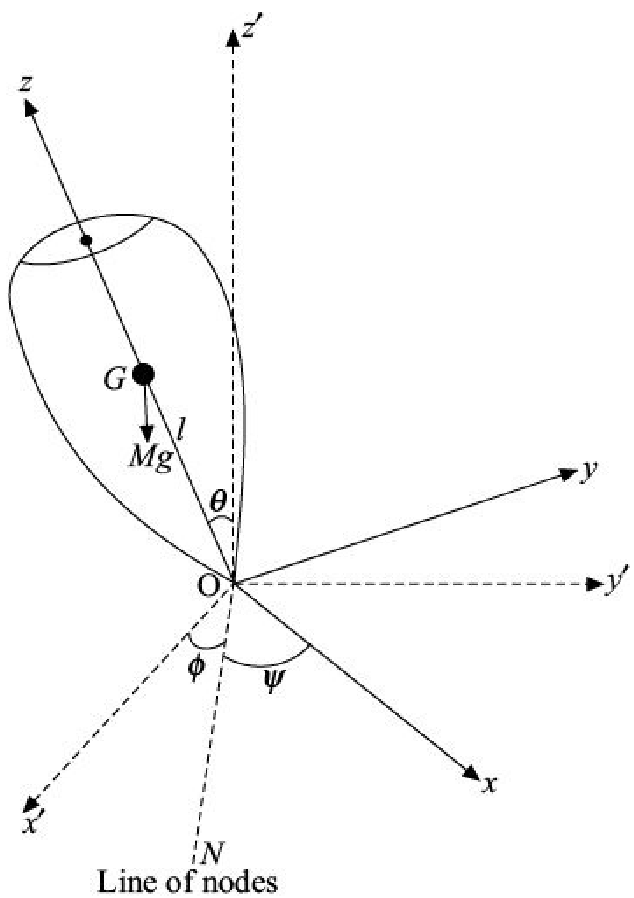


Fig. 8.6 Heavy symmetric top with one point fixed.

Substituting Eq. (8.90) in Eq. (8.89) we get

$$T = \frac{1}{2} I \, (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos\theta)^2 \tag{8.91}$$

The height of the centre of gravity from the point of support is $l \cos \theta$ and therefore the potential energy

$$V = Mg\, l \cos\theta \tag{8.92}$$

where M is the mass of the top. Taking θ, ϕ and ψ as the generalized co-ordinates, the Lagrangian $L = T - V$ takes the form

$$L = \frac{1}{2} I (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos\theta)^2 - Mg\, l \cos\theta \tag{8.93}$$

The generalized co-ordinates ϕ and ψ do not appear explicitly in the Lagrangian and hence they are cyclic co-ordinates. Therefore, the corresponding generalized momenta are constants of motion. That is,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I \dot{\phi} \sin^2 \theta + I_3 (\dot{\psi} + \dot{\phi} \cos\theta) \cos\theta = \text{constant} \tag{8.94}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos\theta) = \text{constant} \tag{8.95}$$

Here, p_f is the angular momentum due to the angular rotation of f about the z-axis and p_y is that due to the angular rotation of y about the z-axis. These are the two first integrals of motion. Another first integral is the total energy *E:*

$$E = T + V = \frac{1}{2} I(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi} \cos\theta)^2 + Mg\,l \cos\theta \quad (8.96)$$

Equations (8.94) and (8.95) can now be solved for $\dot{\phi}$ and $\dot{\psi}$ in terms of θ. Using Equation (8.95) to replace $I_3\,(\dot{\psi} + \dot{\phi} \cos\theta)$ in Eq. (8.94), we get

$$I \dot{\phi} \sin^2 \theta + p_\psi \cos\theta = p_\phi$$

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I \sin^2 \theta} \quad (8.97)$$

Substituting this value of $\dot{\phi}$ in Eq. (8.95)

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos\theta)\cos\theta}{I \sin^2 \theta} \quad (8.98)$$

Substituting Eq. (8.97) to replace $\dot{\phi}$ and Eq. (8.95) to replace $(\dot{\psi} + \dot{\phi} \cos\theta)$ in Eq. (8.96), we get

$$E - \frac{p_\psi^2}{2I_3} = \frac{1}{2} I\, \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I \sin^2 \theta} + Mgl \cos\theta \quad (8.99)$$

Thus, the problem is reduced to motion with one degree of freedom. Replacing the constant on the left side of Eq. (8.99) by *E*, we have

$$E' = \frac{1}{2} I\, \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I \sin^2\theta} + Mgl\cos\theta \qquad (8.100)$$

$$E' = \frac{1}{2} I \dot{\theta}^2 + V'(\theta) \qquad (8.101)$$

where,

$$V'(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I \sin^2\theta} + Mgl\cos\theta \qquad (8.102)$$

$$E' = E - \frac{p_\psi^2}{2I_3} \qquad (8.103)$$

From Eq. (8.101), we have

$$\dot{\theta} = \sqrt{\frac{2(E' - V')}{I}} \qquad (8.104)$$

Integrating

$$t(\theta) = \int \frac{d\theta}{\sqrt{(2/I)\,(E' - V')}} \qquad (8.105)$$

Integration of Eq. (8.105) involves elliptic integrals and the procedure is very complicated. The general feature of the motion can be understood without performing the integration.

The plot of the effective potential $V'(\theta)$ versus q for the physically acceptable range of $0 \le \theta \le \pi$ is given in Fig. 8.7. From Eq. (8.105), it is obvious that the motion will be limited to the case $E' > V'$. For any energy value $E' = E_1'$ the motion is limited between two extreme values q_1 and q_2. This implies that the angle that the symmetry axis Oz can make with the vertical is limited to $\theta_1 \le \theta \le \theta_2$. In other words, the symmetry axis Oz of the top will be bobbing back and forth between two right circular cones of half angles q_1 and q_2 while precessing with the angular velocity $\dot{\phi}$, Eq. (8.97), about Oz. Such a bobbing back and forth motion is called **nutation**. If $\dot{\phi}$ given by Eq. (8.97) does not change sign as q varies between q_1 and q_2, the path described by the projection of the symmetry axis on a unit sphere with the centre at the origin is shown in Fig. 8.8(a). If $\dot{\phi}$ does change sign between the limiting values of q, the precessional angular velocity must have opposite signs at $q = q_1$ and $q = q_2$. In this situation the nutational-precessional symmetry axis describes loops as shown in Fig. 8.8(b). If vanishes at one of the limiting values of q, say q_1, the resulting motion is cusplike as shown in Fig. 8.8 (c).

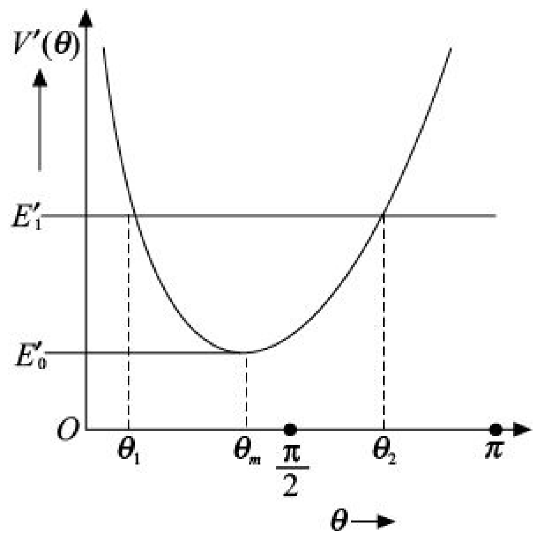


Fig. 8.7 Plot of effective potential $V'(\theta)$ versus q for a heavy symmetrical top.

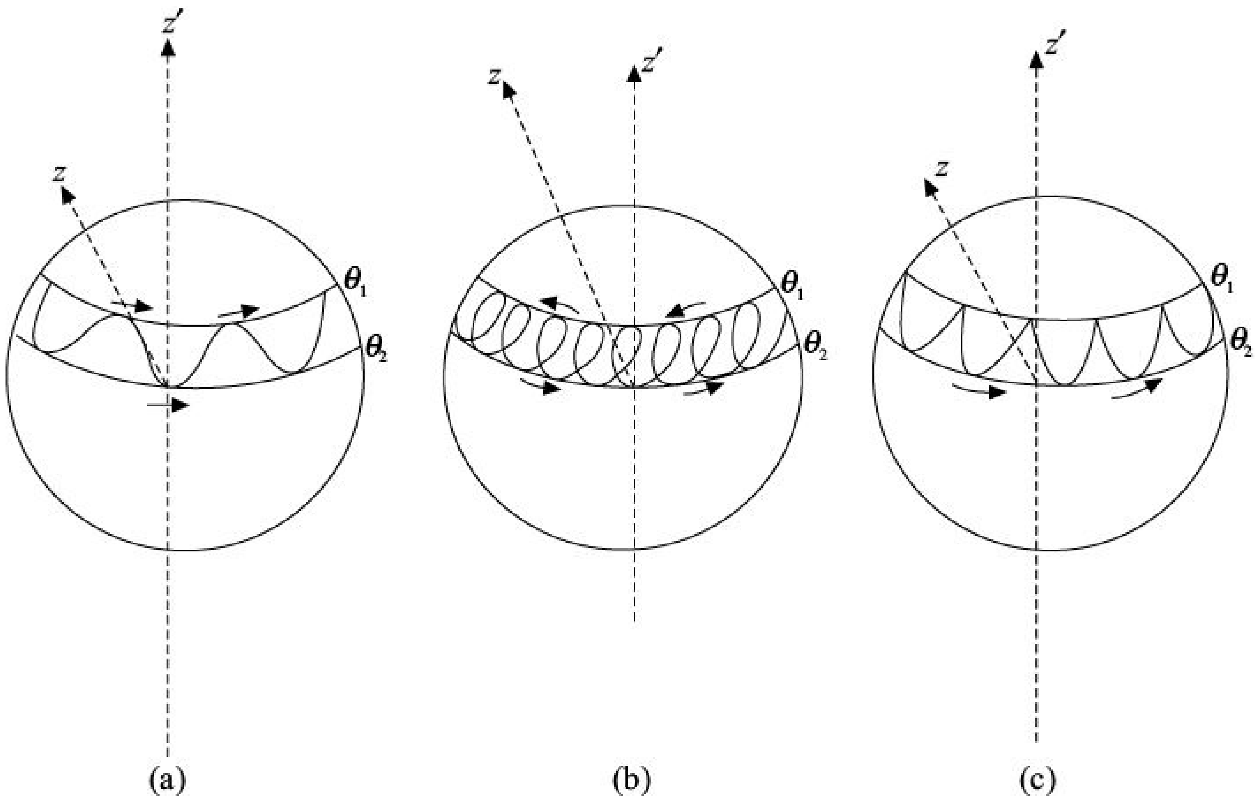


Fig. 8.8 The motion of symmetry axis between $\theta_1 \le \theta \le \theta_2$ projected on a unit sphere in a fixed system: (a) $\dot{\phi}$ never changes sign; (b) $\dot{\phi}$ changes sign at $\theta > \theta_1$; (c) $\dot{\phi}$ vanishes at $\theta = \theta_1$.

Next, we shall consider the case of precession without nutation. If the energy is such that $E' = E'_0 = V'_{\min}$, the value of q is limited to a single value of θ_m. The resulting motion is a steady or pure precession without nutation about Oz'. The steady precession at the fixed angle of inclination of θ_m is possible only if the angular velocity of the spinning top

$$\omega_3 \ge \frac{2}{I_3}\sqrt{Mg\,l\,\cos\theta_m} \tag{8.106}$$

This result can be obtained after obtaining the value of θ_m by setting

$$\left.\frac{\partial V'}{\partial \theta}\right|_{\theta=\theta_m} = 0 \tag{8.107}$$

For the given value of q_m, the precessional angular velocity $\dot{\phi}_m$ has two possible values, one giving rise to a fast precession and the other a slow precession. Slow precession is the one usually observed.

If a top starts spinning sufficiently fast and with its axis vertical, it will remain steady in the upright position for a while. This condition is called **sleeping** and the top is said to be a **sleeping top**. This corresponds to the constant value $q = 0$. The criterion for stability of the sleeping top is given by

$$\omega_3 \ge \frac{2}{I_3}\sqrt{MglI} \tag{8.108}$$

Friction gradually slows down the top, and it starts undergoing a nutation and topples over eventually.

WORKED EXAMPLES

Example 8.1 Find the moments and products of inertia of a homogeneous cube of side a for an origin at one corner, with axes directed along the edges.

Solution: $$I_{xx} = \int_V \rho(y^2 + z^2)\, dV = \rho\int_0^a dx \int_0^a \left[\int_0^a (y^2 + z^2)\, dy\right] dz$$

$$= \rho a \int_0^a \left(\frac{a^3}{3} + az^2\right) dz = \rho a \left(\frac{a^4}{3} + \frac{a^4}{3}\right)$$

$$= \frac{2}{3}\rho a^5 = \rho a^3\, \frac{2}{3} a^2 = \frac{2}{3} M a^2$$

where M is the mass of the cube. By similar arguments

$$I_{yy} = I_{zz} = \frac{2}{3} M a^2$$

$$I_{xy} = -\int_V \rho xy\, dV = -\rho\int_0^a dz \int_0^a \left[\int_0^a xy\, dx\right] dy$$

$$= -\rho a \int_0^a \frac{a^2}{2} y\, dy = -\rho \frac{a^5}{4} = -\frac{1}{4} M a^2$$

$$I = \begin{pmatrix} \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{pmatrix}$$

Example 8.2 Find the principal axes and the principal moments of inertia for a cube of mass M and sides a for an origin at one corner.

Solution: From Example 8.1

$$I_{11} = I_{22} = I_{33} = \frac{2}{3} M a^2 \qquad I_{12} = I_{13} = I_{23} = -\frac{1}{4} M a^2$$

Obviously the specified axes are not principal axes. To find the principal axes we have to solve the determinant

$$\begin{vmatrix} \frac{2}{3}Ma^2 - I & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - I & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - I \end{vmatrix} = 0$$

Subtracting the first row from the second

$$\begin{vmatrix} \frac{2}{3}Ma^2 - I & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{11}{12}Ma^2 + I & \frac{11}{12}Ma^2 - I & 0 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 - I \end{vmatrix} = 0$$

Factoring $\left(\frac{11}{12}Ma^2 - I\right)$ from the second row and then expanding the determinant, we have

$$\left(\frac{11}{12}Ma^2 - I\right)\left[\left(\frac{2}{3}Ma^2 - I\right)^2 - \frac{1}{8}(Ma^2)^2 - \frac{1}{4}Ma^2\left(\frac{2}{3}Ma^2 - I\right)\right] = 0$$

Solving, we get the principal moments of inertia:

$$I_1 = I_2 = \frac{11}{12}Ma^2 \qquad I_3 = \frac{1}{6}Ma^2$$

The degeneracy $I_1 = I_2$ suggests that the 3rd axis corresponds to an axis of symmetry. To find that axis, substitute $I = I_3$ in Eq. (8.33) and solve for the ratio $\omega_1 : \omega_2 : \omega_3$

$$\left(\frac{2}{3} - \frac{1}{6}\right)\omega_1 - \frac{1}{4}\omega_2 - \frac{1}{4}\omega_3 = 0$$

$$-\frac{1}{4}\omega_1 + \left(\frac{2}{3} - \frac{1}{6}\right)\omega_2 - \frac{1}{4}\omega_3 = 0$$

$$-\frac{1}{4}\omega_1 - \frac{1}{4}\omega_2 + \left(\frac{2}{3} - \frac{1}{6}\right)\omega_3 = 0$$

On simplification, we get

$$2\omega_1 - \omega_2 - \omega_3 = 0$$

$$-\omega_1 + 2\omega_2 - \omega_3 = 0$$

$$-\omega_1 - \omega_2 + 2\omega_3 = 0$$

Further simplification gives

$$\omega_1 = \omega_2 = \omega_3$$

That is, when the cube is rotating about this principal axis, the projections of w on the three co-ordinate axes are equal. Hence, the principal axis associated with I_3 coincides with the main diagonal of the cube. The orientation of the principal axes associated with I_1 and I_2 are arbitrary: they have to lie in a plane perpendicular to the diagonal of the cube.

Example 8.3 Obtain the principal moments of inertia of a cube of side *a* and mass *M* by rotating the cube about the co-ordinate axes of a co-ordinate system with one corner as the origin and the axes directed along the sides of the cube.

Solution: The inertia tensor of a cube of side *a* and mass *M* with one corner as the origin and the axes directed along the sides is given by Example 8.1.

$$I = Ma^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

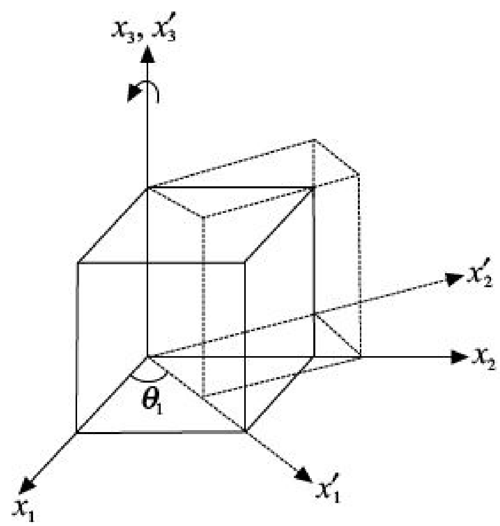


Fig. 8.9 A cube of side *a* with one corner as the origin and the axes along the sides of the cube.

The diagonal of the cube is a principal axis. We have to rotate the cube in such a way that the x_1 axis in Fig. 8.9 coincides with the diagonal of the cube. The length of the face diagonal of a cube $= \sqrt{2}a$ and that of the body diagonal is $\sqrt{3}a.$ The above can be achieved by two successive rotations: (i) rotation through an angle of $\theta_1 = 45°$ about the x_3 axis; and (ii) rotation about x_2' through an angle $\theta_2 = \cos^{-1}\sqrt{2/3}.$ The rotation matrix corresponding to the first rotation is

$$A_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix corresponding to the second rotation is

$$A_2 = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$$

The total rotation is given by matrix A:

$$A = A_2\, A_1 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$I' = AI\tilde{A} = Ma^2 \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Example 8.4 Consider a dumb-bell formed by two point masses *m* at the ends of a massless rod of length 2*a*. It is constrained to rotate with constant angular velocity w about an axis that makes an angle *a* with the rod. Calculate the angular momentum and the torque that is applied to the system.

Solution: Fig. 8.10 shows the dumb-bell rotating with angular velocity w about an axis *AOA* in the inertial co-ordinate system.

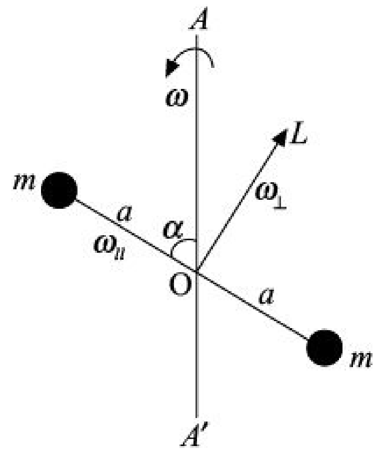


Fig. 8.10 Dumb-bell rotating about an axis that makes an angle *a* with the rod.

Resolving w into components perpendicular and parallel to the dumb-bell axis

$$\omega_{ll} = \omega \cos\alpha \qquad \omega_{\perp} = \omega \sin\alpha$$

The component ω_{ll} produces no angular momentum as the masses are point particles. The moment of inertia about the direction of $\omega_{\perp} = 2ma^2$. The magnitude of the angular momentum vector **L** is

$$|\mathbf{L}| = I\omega_{\perp} = 2ma^2\omega\sin\alpha$$

The angular momentum **L** points in the direction of $\omega_{\perp}$. As the dumb-bell rotates, **L** swings round the rod and the tip traces a circle. By definition, torque

$$\mathbf{N} = \frac{d\mathbf{L}}{dt}$$

Using Eq. (8.53)

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L}$$

$$|\mathbf{N}| = \omega L \sin(90 - \alpha) = \omega L \cos\alpha$$

Substituting the value of $|\mathbf{L}|$

$$|\mathbf{N}| = 2ma^2\omega^2 \sin\alpha \cos\alpha = ma^2\omega^2 \sin 2\alpha$$

Example 8.5 Consider a particle falling freely from a height *h* at latitude *l*. Find its deflection from the vertical due to Coriolis force.

Solution: Let *P* be a point on the surface of earth at latitude *l*. A particle falls vertically from a height *h* above *P*. The velocity of the body is almost vertical, say along the *z*-axis. The angular velocity w of the earth is in the north-south vertical plane or *yz* plane. The Coriolis force –2*m*w $\mathbf{v_r}$ will be in the east-west direction, which will be the deflecting force. Thus, in the northern hemisphere a freely falling body will be deflected to the east, which is taken as the *x*-axis. The equation of motion in the *x*-direction is

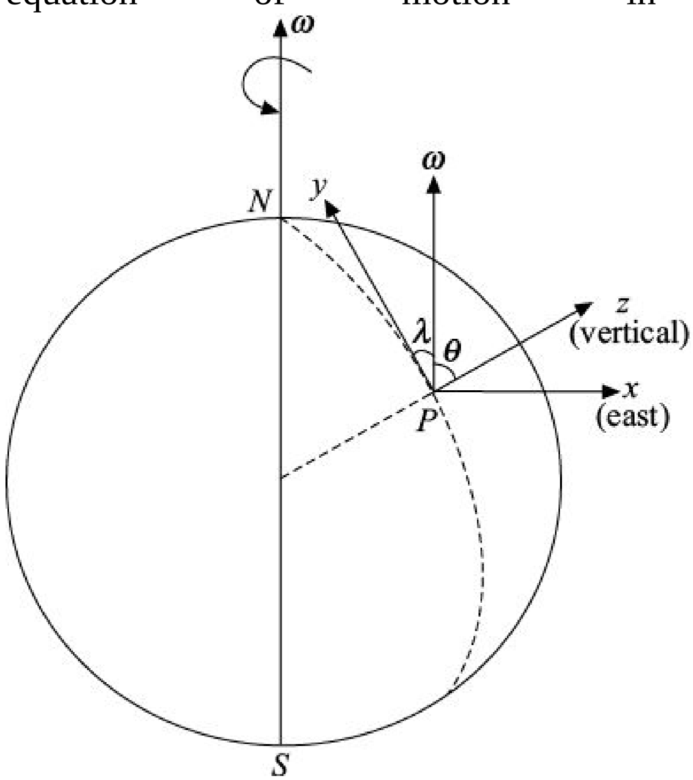


Fig. 8.11 A particle falling freely from a height h at latitude.

$$m\frac{d^2x}{dt^2} = -2m(\boldsymbol{\omega} \times \mathbf{v}_r)_x$$

$$\omega_z = \omega\cos\theta = \omega\cos(90 - \lambda) = \omega\sin\lambda$$

$$\omega_y = \omega\sin\theta = \omega\cos\lambda \qquad \omega_x = 0$$

The components of *w* are (0, *w* cos *l*, *w* sin *l*). The components of *g* are (0, 0, – *g*). Since the Coriolis force is very weak compared to *g*, the *x* and *y* components of the velocity are approximately zero. Hence, the components of the velocity are
(0, 0, – gt). Then

$$\boldsymbol{\omega} \times \mathbf{v}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix}$$

$$(\boldsymbol{\omega} \times \mathbf{v}_r)_x = \omega\cos\lambda(-gt) = -g\omega t\cos\lambda$$

The equation of motion now reduces to $\frac{d^2x}{dt^2} = 2g\omega t\cos\lambda$

Integrating twice with the initial conditions $t = 0$, $\dot{x} = 0$ and $x = 0$, we get

$$x = \frac{1}{3}\omega g\cos\lambda\, t^3$$

Since, $$h = \frac{1}{2}gt^2 \qquad \text{or} \qquad t = \sqrt{\frac{2h}{g}}$$

$$x = \frac{1}{3}\omega g\left(\frac{2h}{g}\right)^{3/2}\cos\lambda$$

That is, there is an eastward deflection given by x when the particle falls freely from a height h at latitude λ.

That is, there is an eastward deflection given by *x* when the particle falls freely from a height *h* at latitude *l*.

Example 8.6 A body can rotate freely about the principal axis corresponding to the principal moment of inertia I_3. If it is given a small displacement, show that the rotation will be oscillatory if I_3 is either the largest or the smallest of the three principal moments of inertia.

Solution: Since the displacement is small, we may take w_1 and w_2 as small and the product $w_1 w_2$ may be neglected. From the third equation of Eq. (8.72) we get

$$\dot{\omega}_3 = 0 \qquad \text{or} \qquad \omega_3 = \text{constant}$$

From the first equation of Eq. (8.72), we have

$$\ddot{\omega}_1 = \frac{\omega_3 (I_2 - I_3)}{I_1} \dot{\omega}_2$$

Substituting the value of $\dot{\omega}_2$ from the second equation of Eq. (8.72)

$$\ddot{\omega}_1 = \left[\frac{(I_3 - I_2)(I_1 - I_3)}{I_1 I_2} \omega_3^2 \right] \omega_1$$

$$\ddot{\omega}_1 = k^2 \omega_1 \qquad k^2 = \text{constant}$$

As ω_3^2 and $I_1 I_2$ are positive constants, the nature of the solution is decided by the product $(I_3 - I_2)(I_1 - I_3)$. If $I_3 > I_1$ and $I_3 > I_2$ or $I_3 < I_1$ and $I_3 < I_2$, the equation reduces to

$$\ddot{\omega}_1 = -k^2 \omega_1$$

and the solution for w_1 will be oscillatory. On the other hand, if $I_1 > I_3 > I_2$ or $I_1 < I_3 < I_2$, the equation becomes $\ddot{\omega}_1 = k^2 \omega_1$

the solution will be exponentially increasing with time. Similar arguments hold good for w_2 also. Hence, the rotation will be oscillatory if I_3 is either the largest or the smallest of the three principal moments of inertia.

Example 8.7 A body moves about a point O under no force. The principal moments of inertia at O being 3*A*, 5*A* and 6*A*. Initially the angular velocity has components $w_1 = w$, $w_2 = 0$ and $w_3 = w$ about the corresponding principal axes. Show that at time *t*

$$\omega_2 = \frac{3\omega}{\sqrt{5}} \tan \frac{\omega t}{\sqrt{5}} \qquad \text{if} \qquad \int \frac{dx}{p^2 - x^2} = \frac{1}{p} \tan \text{h}^{-1} \left(\frac{x}{p} \right)$$

Solution: In the torque-free case, the Euler's equations are

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) \tag{i}$$

$$I_2 \dot{\omega}_2 = \omega_1 \omega_3 (I_3 - I_1) \tag{ii}$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) \tag{iii}$$

Replacing the principal moments of inertia I_1, I_2, I_3 by $3A$, $5A$ and $6A$, respectively

$$3\dot{\omega}_1 = -\omega_2 \omega_3 \tag{iv}$$

$$5\dot{\omega}_2 = 3\omega_3 \omega_1 \tag{v}$$

$$6\dot{\omega}_3 = -2\omega_1 \omega_2 \tag{vi}$$

Multiplying Eq. (iv) by $3\omega_1$ and (v) by ω_2 and adding the two

$$9\omega_1 \dot{\omega}_1 + 5\omega_2 \dot{\omega}_2 = 0$$

Integrating and applying the initial conditions

$$9\omega_1^2 + 5\omega_2^2 = \text{constant}$$

$$9\omega_1^2 + 5\omega_2^2 = 9\omega^2 \tag{vii}$$

Similarly from Eqs. (iv) and (vi)

$$\omega_1^2 = \omega_3^2 \tag{viii}$$

Using Eqs. (viii), (v) and (vii), we have

$$5\dot{\omega}_2 = 3\omega_1^2 = 3\omega^2 - \frac{5\omega_2^2}{3} \qquad \text{or} \qquad \dot{\omega}_2 = \frac{9\omega^2 - 5\omega_2^2}{15}$$

Integrating

$$t = 15\int \frac{d\omega_2}{9\omega^2 - 5\omega_2^2} = 3\int \frac{d\omega_2}{(9/5)\omega^2 - \omega_2^2}$$

$$= \frac{\sqrt{5}}{\omega}\tanh^{-1}\left(\frac{\sqrt{5}\,\omega_2}{3\omega}\right)$$

$$\omega_2 = \frac{3\omega}{\sqrt{5}}\tanh\left(\frac{\omega t}{\sqrt{5}}\right)$$

Example 8.8 If w_3 is the angular velocity of a freely rotating symmetric top about its symmetry axis, show that the symmetry axis rotates about the space-fixed *z*-axis with angular frequency $\dot{\phi} = \frac{(2I_1 - I_3)\omega_3}{I_1 \cos\theta}$, where *q* and *f* are Euler's angles.

Solution: From the third equation of Eq.(8.49), we have $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$

(i) In the force-free motion of a symmetric top we have seen that the angular velocity vector w of the top precesses in a cone about the body sysmmetry axis with an angular frequency *k* given by $k = \frac{(I_3 - I_1)\omega_3}{I_1}$ (ii) This angular frequency is the same as $\dot{\psi}$ which is also directed along the symmetry axis. Substituting this value of $\dot{\psi}$ in the expression for w_3 and simplifying,

we get $\dot{\phi} = \frac{(2I_1 - I_3)\,\omega_3}{I_1 \cos\theta}$

Example 8.9 In the absence of external torque on a body, prove that (i) the kinetic energy is constant; (ii) the magnitude of the square of the angular momentum (L^2) is constant.

Solution: (i) Multiplying the first of Eq. (8.72) by w_1, the second by w_2 and the third by w_3, and adding, we get

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0$$

$$\frac{1}{2}\frac{d}{dt}[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2] = 0$$

The quantity inside the square bracket is kinetic energy $2T$, that is

$$\frac{d}{dt}(T) = 0 \qquad \text{or} \qquad T \text{ is a constant.}$$

(ii)
$$L^2 = (I_1\omega_1 + I_2\omega_2 + I_3\omega_3)\cdot(I_1\omega_1 + I_2\omega_2 + I_3\omega_3)$$

$$= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Multiplying the first of Eq. (8.72) by $I_1\omega_1$, the second by $I_2\omega_2$ and the third by $I_3\omega_3$ and adding, we get

$$I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3 = 0$$

$$\frac{1}{2}\frac{d}{dt}[I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2] = 0$$

$$\frac{d}{dt}L^2 = 0$$

$$L^2 = \text{constant of motion.}$$

REVIEW QUESTIONS

1. In general, the angular momentum vector **L** is not necessarily always in the same direction as the instantaneous axis of rotation. Substantiate.
2. What are moments of inertia and products of inertia?
3. What is Poinsot's ellipsoid of inertia?
4. Express the rotational kinetic energy of a body in terms of inertia tensor and angular velocity.
5. What are principal axes and principal moments of inertia?
6. When do you say a body is a symmetric top? Give an example. Distinguish between prolate and oblate symmetric tops.
7. If the rotation axis of a body is in the direction of a principal axis, show that the angular velocity vector and angular momentum will be in the same direction.
8. If the moments of inertia and products of inertia of a body with respect to an

arbitrary co-ordinate system are known, how do you find out the principal moments of inertia in a principal axes system?

9. What are Euler's angles?
10. State and explain Euler's geometrical equations.
11. An infinitesimal rotation can be represented by a vector along the instantaneous axis of rotation. Substantiate.
12. Express the inertial acceleration of a particle of mass *m* in terms of its acceleration relative to a rotating frame.
13. What are centrifugal and Coriolis forces?
14. What do you understand by nutation?
15. In the force-free motion of a rigid body, distinguish between body cone and space cone.
16. Explain the precessional motion with and without nutation in the case of a spinning heavy symmetric top.

PROBLEMS

1. A rigid body of mass *M* is suspended and allowed to swing freely under its own weight about a fixed horizontal axis of rotation. Obtain an expression for the frequency of oscillation and find the length of an equivalent (in frequency) simple pendulum.
2. Find the moment of inertia tensor for the configuration. in which point masses of 1, 2, 3 and 4 units are located at (1, 0, 0), (1, 1, 0), (1, 1, 1) and (1, 1, –1).
3. Find the moments and products of inertia for a homogeneous rectangular parallelopiped of mass *M* with edges *a*, *b*, *c* with co-ordinate axes along the edges and the origin located at one corner.
4. A rigid body is rotating about the *x*-axis. Find: (i) the angular momentum vector **L**; (ii) the condition for **L** and w to be parallel; (iii) the kinetic energy of the body under that condition.
5. Find the moments and products of inertia for a rectangular parallelopiped of mass *M* with edges *a*, *b*, *c* with its origin at the centre of mass and axes parallel to the three edges.
6. In the principal axes system, express the rotational kinetic energy of a rigid symmetric body ($I_1 = I_2 = I$) in terms of Euler's angles.
7. A rigid body is rotating under the influence of an external torque **N** acting on it. If w is the angular velocity and *T* is its kinetic energy, show that in the principal axes system.
8. If a rectangular parallellopiped with its edges *a*, *a*, *b* rotates about its centre of

gravity under no forces, prove that its angular velocity about one principal axis is constant. Also prove that the motion is periodic about the other two axes.

[Hint: the values of the principal moment's of inertia are $I_1 = I_2 = m(a^2 + b^2)/12$

$I_3 = m(a^2 + a^2)/12$.]

9. A body is dropped from rest at a height of 300 m above the surface of the earth at a latitude of 45°. Find the magnitude of deflection due to Coriolis force when the body touches the earth.

9

Theory of Small Oscillations

The theory of small oscillations about the equilibrium position is of importance in molecular spectra, acoustics, vibrations of atoms in solids, vibrations of coupled mechanical systems and coupled electrical circuits. If the displacement from the stable equilibrium conditions are small, the motion can be described as that of a system of coupled linear harmonic oscillators with each generalized coordinate expressed as a function of the different frequencies of vibrations of the system. The problem can be simplified further by a transformation of the generalized coordinates to another set of coordinates, each of which undergoes periodic changes with a well-defined single frequency. In this chapter we develop a theory of small oscillations based on Lagrangian formulation.

9.1 EQUILIBRIUM AND POTENTIAL ENERGY

To understand the motion of a system in the neighbourhood of stable equilibrium, it is essential that we should know the relation between potential energy and equilibrium. Let us consider a conservative system having *n* degrees of freedom with generalized coordinates q_1, q_2..., q_n. Since the system is conservative, the potential energy *V* is a function of the generalized coordinates

$$V = V\left(q_1, q_2, \ldots, q_n\right) \qquad (9.1)$$

The system is said to be in equilibrium if the generalized forces acting on the system vanish:

$$Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \qquad (9.2)$$

An equilibrium position of the system is said to be **stable** if, after a small disturbance, the system does return to its original configuration. If the system does not return to its original configuration it is in an **unstable equilibrium**. On the other hand, if the system is displaced and it has no tendency to move toward or away from the equilibrium configuration, the system is said to be in **neutral equilibrium**.

Figure 9.1 gives the form of a potential function *V* versus *q* curve. At points *A* and *B*, $(\partial V / \partial q) = 0$ and therefore they are equilibrium positions. Let the potential and kinetic energies

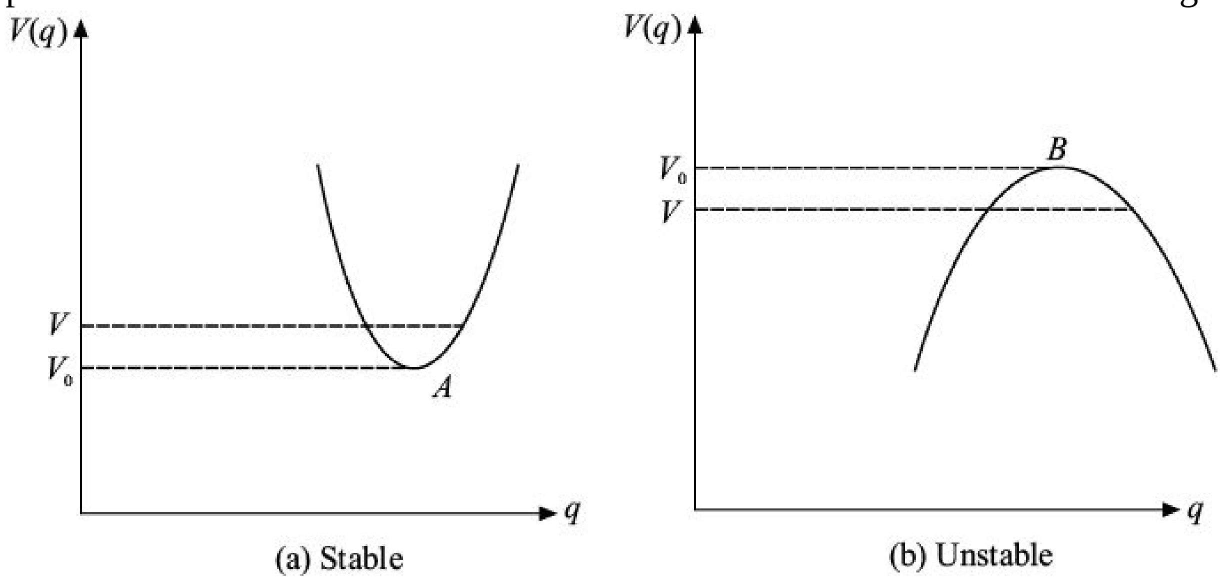


Fig. 9.1 Form of the potential energy curve at equilibrium.

of a system in the equilibrium position be V_0 and T_0, respectively. Suppose the system is given a small displacement and the potential and kinetic energies at any subsequent time be *V* and *T*. By the law of conservation of energy $T_0 + V_0 = T + V$

$T - T_0 = -(V - V_0)$ (9.3) Assume that the system is in equilibrium corresponding to the configuration at *A*, where the potential energy V_0 is minimum. Any displacement from this equilibrium position will lead to a potential energy $V > V_0$. Then $V - V_0$ is a positive quantity and from Eq. (9.3), $T - T_0$ is negative or $T < T_0$. Since *T* decreases with displacement, the velocity decreases and finally comes to zero; then it will start coming back to the equilibrium configuration. Thus, the system will be in stable equilibrium. However, if *V* decreases as a result of a small displacement from an equilibrium position, $T - T_0$ will be positive and velocity

increases indefinitely, corresponding to an unstable motion. This situation corresponds to position
B in Fig. 9.1.

Thus, for small displacements, the condition for stable equilibrium is that the potential energy V_0 is minimum at the equilibrium configuration.

9.2 THEORY OF SMALL OSCILLATIONS

Consider a conservative system having *n* degrees of freedom, described by a set of *n* generalized coordinates q_1, q_2,..., q_n. The system has a stable equilibrium corresponding to the minimum of potential energy V_0. Let us assume that the generalized coordinates are measured with respect to this stable equilibrium position. Expanding the potential $V(q_1, q_2,..., q_n)$ of the system about the equilibrium point in a Taylor series, we have

$$V\ (q_1, q_2,..., q_n) = V_0 + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum_i \sum_j \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i\, q_j + \cdots \quad (9.4)$$

The first term is the potential energy at the equilibrium position which is a constant and may be taken as zero. The second term vanishes, since at the equilibrium position $(\partial V / \partial q_i)_0 = 0.$ Neglecting higher terms

$$V = \frac{1}{2}\sum_i \sum_j V_{ij}\ q_i\, q_j \quad (9.5)$$

where,

$$V_{ij} = \left(\frac{\partial^2 V}{\partial q_i\, \partial q_j}\right)_0 \quad (9.6)$$

It is obvious that V_{ij}'s are symmetric, since the second derivatives are evaluated at the equilibrium position and the order of differentiation is immaterial. The diagonal elements represent the force constant of the restoring force acting on the particle. Since *V* is measured from its minimum value and this minimum is taken as zero, V > 0.

If the transformation equations defining the generalized coordinates do not depend explicitly on time, the kinetic energy is a quadratic function of the generalized velocities. That is,

$$T = \frac{1}{2}\sum_i \sum_j m_{ij}\ \dot{q}_i\ \dot{q}_j \qquad (9.7)$$

where the m_{ij}'s are in general functions of the generalized coordinates and contain the masses. Expanding m_{ij} into a Taylor series about the equilibrium values of q_i's and neglecting terms beyond the constant values of m_{ij} at the equilibrium position $m_{ij} = (m_{ij})_0$ (9.8) Designating the constant values of $(m_{ij})_0$ by the constant G_{ij}'s

$$T = \frac{1}{2}\sum_i \sum_j G_{ij}\ \dot{q}_i \dot{q}_j \qquad (9.9)$$

Again it is obvious that the constants G_{ij} must be symmetric, since the individual terms are unaffected by an interchange of indices. For the case $i = 1, 2$ and $j = 1, 2$

$$T = \frac{1}{2} G_{11}\, \dot{q}_1^2 + \frac{1}{2} G_{22}\, \dot{q}_2^2 + G_{12}\, \dot{q}_1 \dot{q}_2$$

Thus, G_{11} is the coefficient of $\left(\frac{1}{2}\right)\dot{q}_1^2$, G_{22} is the coefficient of $\left(\frac{1}{2}\right)\dot{q}_2^2$ and G_{12} is the coefficient of $\dot{q}_1\, \dot{q}_2$.

Now we are in a position to write the Lagrangian of the system:

$$L = T - V = \frac{1}{2}\sum_i \sum_j (G_{ij}\ \dot{q}_i\ \dot{q}_j - V_{ij}\ q_i\ q_j) \qquad (9.10)$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{1}{2}\sum_j G_{ij}\ \dot{q}_j \qquad (9.11)$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

takes the form

$$\sum_j (G_{ij}\ddot{q}_j - V_{ij} q_j) = 0, \quad i = 1, 2, \ldots, n \qquad (9.12)$$

Equation (9.12) is a system of n second order homogeneous differential equations with constant coefficients. Each of the equations involves all the *n* coordinates.

The form of the equations suggests an oscillatory solution of the type

$$q_j = Ca_j e^{-i\omega t} \qquad (9.13)$$

where the amplitude Ca_j is to be determined from initial conditions, while the natural frequency *w* is determined from the system's constant. The factor *C* is a scale factor, which is the same for all coordinates. Substituting Eq. (9.13) into Eq. (9.12), we get

$$\sum_j (V_{ij} - \omega^2 G_{ij})\, a_j = 0, \qquad i = 1, 2, \ldots, n \qquad (9.14)$$

Expanding the equation

$$\begin{aligned}
&(V_{11} - \omega^2 G_{11})\, a_1 + (V_{12} - \omega^2 G_{12})\, a_2 + \ldots + (V_{1n} - \omega^2 G_{1n})\, a_n = 0 \\
&(V_{21} - \omega^2 G_{21}) a_1 + (V_{22} - \omega^2 G_{22}) a_2 + \ldots + (V_{2n} - \omega^2 G_{2n})\, a_n = 0 \qquad (9.15) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&(V_{n1} - \omega^2 G_{n1})\, a_1 + (V_{n2} - \omega^2 G_{n2})\, a_2 + \ldots + (V_{nn} - \omega^2 G_{nn})\, a_n = 0
\end{aligned}$$

This set of *n* linear homogeneous equations for the *a*'s will have a solution only if the determinant of the coefficients vanishes:

$$\begin{vmatrix}
V_{11} - \omega^2 G_{11} & V_{12} - \omega^2 G_{12} \ldots\ldots\ldots & V_{1n} - \omega^2 G_{1n} \\
V_{21} - \omega^2 G_{21} & V_{22} - \omega^2 G_{22} \ldots\ldots\ldots & V_{2n} - \omega^2 G_{2n} \\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\
V_{n1} - \omega^2 G_{n1} & V_{n2} - \omega^2 G_{n2} \ldots\ldots\ldots & V_{nn} - \omega^2 G_{nn}
\end{vmatrix} = 0 \qquad (9.16)$$

This equation is of *n*th degree in w^2, and the roots give the frequencies for which Eq. (9.13) represents a correct solution of Eq. (9.12). That is, the equations of motion will be satisfied by an oscillatory solution of the type given in Eq. (9.13), not merely for one frequency but in general for a set of *n* frequencies w_p. These frequencies are often called the **frequencies of free vibration** or the **resonant frequencies** of the system. A complete solution of the equations of motion therefore involves a superposition of oscillations with all the allowed frequencies: $$q_j = \sum_p C_p a_{jp} e^{-i\omega_p t} \qquad (9.17)$$

Each of the coordinates is dependent on all the frequencies and none is a periodic function involving only one resonant frequency.

To determine the amplitudes (a_j's), each value of w_p is substituted separately in Eq. (9.15). By this it is possible to determine (n – 1) coefficients in terms of the nth one. The value of the nth coefficient must be determined arbitrarily.

9.3 NORMAL MODES

As discussed in the previous section, the expression in Eq. (9.17) for the coordinates q_j contains n terms, and each term corresponds to one frequency. It is possible to effect a linear transformation to new generalized coordinates Q_1, Q_2, Q_3... so that one coordinate contains only one frequency. Let the transformation be of the form $$q_j = \sum_r a_{jr} Q_r \qquad (9.18)$$

Next we shall express the potential energy V and the kinetic energy T in terms of the new coordinates, the Q's. In terms of column vectors q and Q, Eq. (9.18) can be written as $q = AQ$ (9.19) where A is a matrix, called the *matrix of eigenvectors,* formed by the eigenvectors (a's). Eq. (9.14) can also be written as the matrix equation

$$Va = \omega^2 Ga \qquad (9.20)$$

The matrix of the eigenvectors A diagonalizes both G and V, G to a unit matrix and V to a matrix whose diagonal elements are the eigenvalues w^2:

$$\tilde{A}GA = 1 \qquad \text{and} \qquad \tilde{A}VA = \omega^2 \qquad (9.21)$$

The potential energy V, given in Eq. (9.5), in matrix form is

$$V = \frac{1}{2}\tilde{q}Vq \tag{9.22}$$

From Eq. (9.19)

$$\tilde{q} = \tilde{Q}\tilde{A} \tag{9.23}$$

Substituting the values of q and $\tilde{q}$ in Eq. (9.22)

$$V = \frac{1}{2}\tilde{Q}\tilde{A}V\,AQ \tag{9.24}$$

Using Eq. (9.21)

$$V = \frac{1}{2}\tilde{Q}\omega^2 Q$$

$$V = \frac{1}{2}\sum_p \omega_p^2 Q_p^2 \tag{9.25}$$

Since the velocities transform in the same way as the co-ordinates, Eq. (9.9) transforms to

$$T = \frac{1}{2}\dot{\tilde{Q}}\tilde{A}\,GA\dot{Q} = \frac{1}{2}\dot{\tilde{Q}}\dot{Q} \tag{9.26}$$

$$T = \frac{1}{2}\sum_p \dot{Q}_p^2 \tag{9.27}$$

It is evident from Eqs. (9.25) and (9.27) that both potential and kinetic energies are homogeneous quadratic functions without any cross terms.

In terms of the new coordinates the Lagrangian

$$L = T - V = \frac{1}{2}\sum_p (\dot{Q}_p^2 - \omega_p^2 Q_p^2) \tag{9.28}$$

Lagrange's equations are

$$\ddot{Q}_p + \omega_p^2 Q_p = 0 \qquad p = 1, 2, 3, \ldots, n \tag{9.29}$$

Its solutions are given by

$$Q_p = C_p e^{-i\omega_p t} \qquad p = 1, 2, \ldots, n \tag{9.30}$$

Thus, each of the new coordinates is a periodic function involving only one of the resonant frequencies. The coordinates Q_1, Q_2,..., Q_n are called **normal coordinates** and w_1, w_2,..., w_n are the corresponding **normal frequencies**. Each normal coordinate corresponds to a vibration with only one frequency. These component vibrations are called **normal modes of vibration**. In each mode, all the particles vibrate with the same frequency and with the same phase. Particles may be exactly out of phase if the *a*'s have opposite sign.

9.4 TWO COUPLED PENDULA

Consider two identical simple pendula of mass *m* and length *l* connected by a massless spring of spring constant *k*. The displacement of the bobs to the right are x_1 and x_2 (see Fig. 9.2), and the corresponding angular displacements are q_1 and q_2. The potential energy when the bob is at the mean position is taken as zero. Angles q_1 and q_2 can be taken as the generalized coordinates.

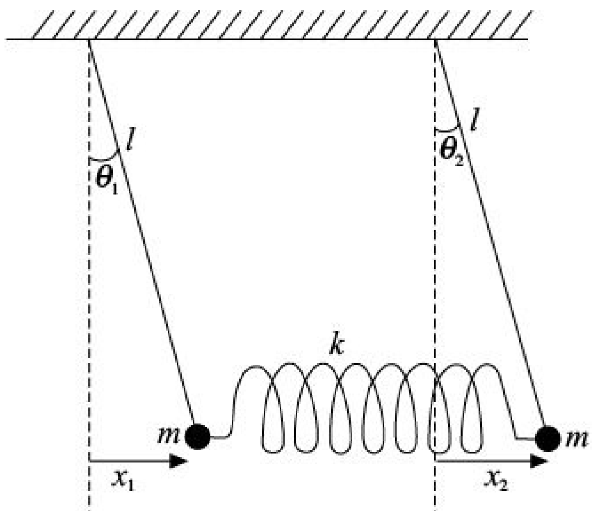


Fig. 9.2 Two simple pendula coupled by a spring.

Resonant Frequencies The potential energy of the system

$$V = mgl\left(1 - \cos\theta_1\right) + mgl\left(1 - \cos\theta_2\right) + \frac{1}{2}k\left(x_2 - x_1\right)^2$$

$$= mgl\left(1 - \cos\theta_1\right) + mgl\left(1 - \cos\theta_2\right) + \frac{1}{2}kl^2\left(\sin\theta_2 - \sin\theta_1\right)^2$$

Since θ is small

$$\cos\theta_1 \cong 1 - \theta_1^2/2 \qquad \cos\theta_2 = 1 - \theta_2^2/2 \qquad \sin\theta_2 - \sin\theta_1 \cong (\theta_2 - \theta_1).$$

Substituting these values and simplifying

$$V = \frac{1}{2}(mgl + kl^2)\,\theta_1^2 + \frac{1}{2}(mgl + kl^2)\,\theta_2^2 - kl^2\,\theta_1\theta_2 \tag{9.31}$$

$$V_{11} = \left(\frac{\partial^2 V}{\partial\theta_1\,\partial\theta_1}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = mgl + kl^2 \quad V_{12} = \left(\frac{\partial^2 V}{\partial\theta_1\,\partial\theta_2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = -\,kl^2 \tag{9.32a}$$

$$V_{22} = \left(\frac{\partial^2 V}{\partial\theta_2\,\partial\theta_2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = mgl + kl^2 \tag{9.32b}$$

Kinetic energy $\qquad T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right)$

$$= \frac{1}{2}m(l\,\dot{\theta}_1)^2 + \frac{1}{2}m(l\,\dot{\theta}_2)^2 \tag{9.33}$$

The elements G_{11} and G_{22} are the coefficients of $\frac{1}{2}\dot{\theta}_1^2$ and $\frac{1}{2}\dot{\theta}_2^2$ respectively. Also, it is obvious that G_{12} and G_{21} are zero. Hence,

$$G_{11} = G_{22} = ml^2 \qquad G_{12} = G_{21} = 0 \tag{9.34}$$

The resonant frequencies can be obtained from Eq. (9.16). The secular determinant is

$$V - \omega^2 G = \begin{vmatrix} mgl + kl^2 - \omega^2 ml^2 & -kl^2 \\ -kl^2 & mgl + kl^2 - \omega^2 ml^2 \end{vmatrix} = 0 \qquad (9.35)$$

$$\left(\frac{mg}{l} + k - \omega^2 m\right)^2 - k^2 = 0$$

$$\left(\frac{mg}{l} + 2k - \omega^2 m\right)\left(\frac{mg}{l} - \omega^2 m\right) = 0 \qquad (9.36)$$

$$\omega^2 = \frac{g}{l} + \frac{2k}{m} \qquad \text{or} \qquad \frac{g}{l}$$

$$\omega_1 = \sqrt{\frac{g}{l}} \qquad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}} \qquad (9.37)$$

One of the resonant frequencies, $\sqrt{g/l}$, is the same as that of a free pendulum of the same length. In the other mode, both the pendula and the spring participate.

Normal Modes Next we shall find the normal modes of the system. Eq. (9.14) gives

$$\sum_{j=1,2} \left(V_{ij} - \omega^2 G_{ij}\right) a_j = 0 \qquad i = 1, 2 \qquad (9.38)$$

Substituting the values of V_{ij} and G_{ij}

$$(mgl + kl^2 - \omega^2 ml^2)\, a_1 - kl^2 a_2 = 0 \qquad (9.39)$$

$$-kl^2 a_1 + (mgl + kl^2 - \omega^2 ml^2)\, a_2 = 0 \qquad (9.40)$$

Substituting $\omega_1^2 = g/l$ in Eqs. (9.39) and (9.40)

$$kl^2 a_1 - kl^2 a_2 = 0 \qquad (9.41)$$

$$-kl^2 a_1 + kl^2 a_2 = 0 \qquad (9.42)$$

From Eqs. (9.41) and (9.42)

$$a_1 = a_2 = \alpha \qquad (9.43)$$

These eigenvectors correspond to the value of $w = w_1$. To get the eigenvectors corresponding to the value of $w = w_2$, substitute the value of w_2 in Eqs. (9.39) and (9.40). We get

$$- kl^2 a_1 - kl^2 a_2 = 0 \tag{9.44}$$

$$-kl^2 a_1 - kl^2 a_2 = 0 \tag{9.45}$$

It is obvious from Eqs. (9.44) and (9.45)

$$a_1 = - a_2 = \beta \tag{9.46}$$

Consequently, the matrix of eigenvectors denoted by A is given by

$$A = \begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix} \tag{9.47}$$

Using the condition given in Eq. (9.21) that $\tilde{A}GA = 1$, we have

$$\begin{pmatrix} \alpha & \alpha \\ \beta & -\beta \end{pmatrix} \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2ml^2\alpha^2 & 0 \\ 0 & 2ml^2\beta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.48}$$

$$2ml^2\alpha^2 = 2ml^2 \ \beta^2 = 1$$

$$\alpha = \beta = \frac{1}{\sqrt{2ml^2}} \tag{9.49}$$

With these values of α and β, the matrix of eigenvectors

$$A = \begin{pmatrix} \frac{1}{\sqrt{2ml^2}} & \frac{1}{\sqrt{2ml^2}} \\ \frac{1}{\sqrt{2ml^2}} & -\frac{1}{\sqrt{2ml^2}} \end{pmatrix} \tag{9.50}$$

From Eq. (9.19)

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2ml^2}} & \frac{1}{\sqrt{2ml^2}} \\ \frac{1}{\sqrt{2ml^2}} & -\frac{1}{\sqrt{2ml^2}} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \tag{9.51}$$

where Q_1 and Q_2 are the normal co-ordinates of the system. Expanding Eq. (9.51)

$$\theta_1 = \frac{Q_1}{\sqrt{2ml^2}} + \frac{Q_2}{\sqrt{2ml^2}} \tag{9.52}$$

$$\theta_2 = \frac{Q_1}{\sqrt{2ml^2}} - \frac{Q_2}{\sqrt{2ml^2}} \tag{9.53}$$

Solving

$$Q_1 = \sqrt{\frac{ml^2}{2}}\left(\theta_1 + \theta_2\right) \tag{9.54}$$

$$Q_2 = \sqrt{\frac{ml^2}{2}}\left(\theta_1 - \theta_2\right) \tag{9.55}$$

Next we shall see the physical meaning of these normal modes. For the Q_1 mode, we take Q_2 = O; therefore $q_1 - q_2 = 0$ or $q_1 = q_2$ (9.56) That is, the two pendula are oscillating in phase. This is the symmetric mode of oscillation shown in Fig. 9.3 (a). For the Q_2 mode, we take $Q_1 = 0$; then $q_1 + q_2 = 0$ or $q_1 = -q_2$ (9.57) That is, the two pendula are oscillating out of phase with each other. This is the antisymmetric mode shown in Fig. 9.3 (b).

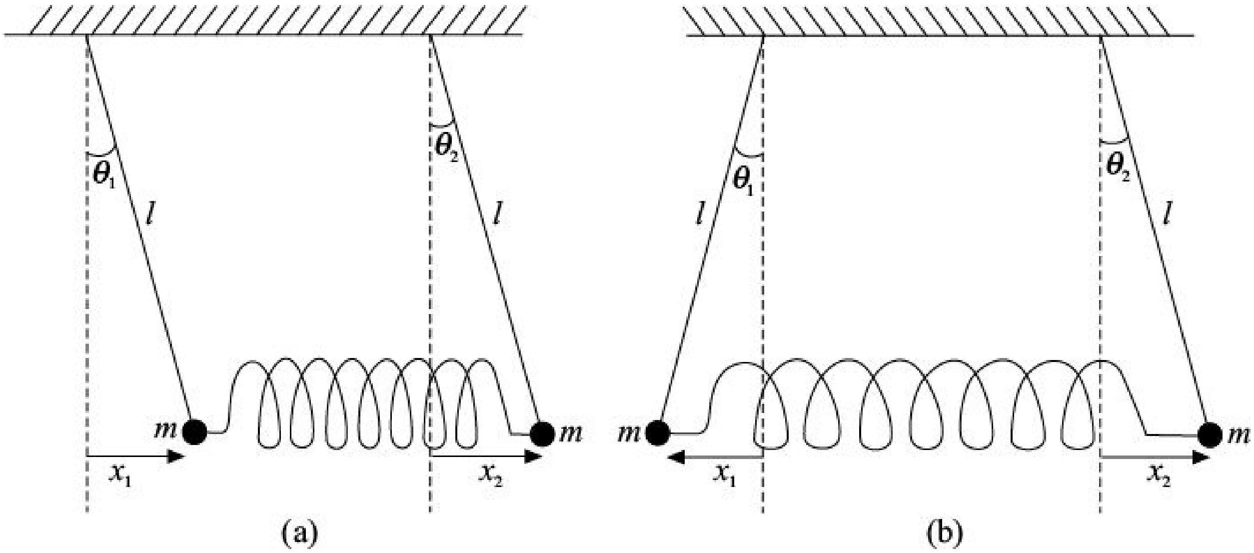


Fig. 9.3 Normal modes of two coupled pendula: (a) Symmetric mode: $q_1 = q_2$, $x_1 = x_2$; (b) Antisymmetric mode: $q_1 = -q_2$, $x_1 = -x_2$.

9.5 LONGITUDINAL VIBRATIONS OF CO_2 MOLECULE

In the CO_2 molecule, the three atoms are in the same straight line. The complicated interatomic potential can be approximated by two springs of force constant k joining the three atoms as shown in Fig. 9.4. The displacement coordinates marking the position of the three atoms are also shown in the figure.

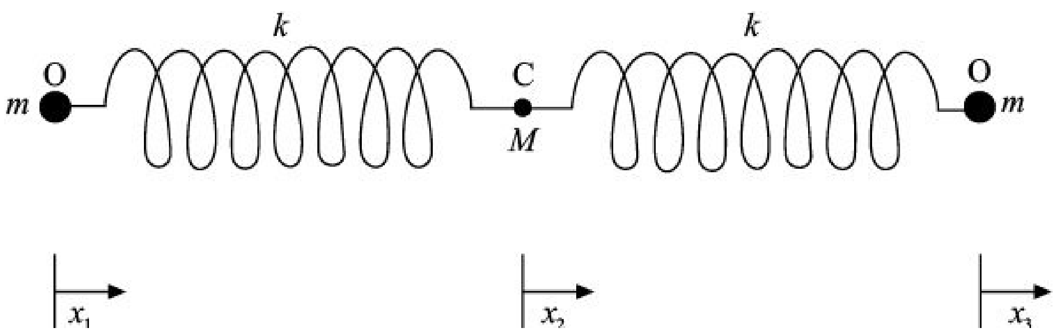


Fig. 9.4 A linear symmetrical CO_2 molecule.

Normal Frequencies The potential energy

$$V = \frac{1}{2}k\,(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2$$

$$= \frac{1}{2}k(x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_3x_2) \tag{9.58}$$

From Eq. (9.6)

$$V_{11} = \left(\frac{\partial^2 V}{\partial x_1^2}\right)_0 = k \qquad V_{12} = \left(\frac{\partial^2 V}{\partial x_1 \partial x_2}\right)_0 = -k$$

$$V_{22} = \left(\frac{\partial^2 V}{\partial x_2^2}\right)_0 = 2k \qquad V_{13} = \left(\frac{\partial^2 V}{\partial x_1 \partial x_3}\right)_0 = 0 \tag{9.59}$$

$$V_{33} = \left(\frac{\partial^2 V}{\partial x_3^2}\right)_0 = k \qquad V_{23} = \left(\frac{\partial^2 V}{\partial x_2 \partial x_3}\right)_0 = -k$$

The kinetic energy

$$T = \frac{1}{2}m\,\dot{x}_1^2 + \frac{1}{2}M\,\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 \tag{9.60}$$

From Eqs. (9.7) and (9.9)

$$G_{11} = G_{33} = m \qquad G_{22} = M$$

$$G_{12} = G_{23} = G_{13} = 0 \tag{9.61}$$

The secular determinant shown in Eq. (9.16) takes the form

$$\begin{vmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{vmatrix} = 0 \tag{9.62}$$

Expanding the determinant and simplifying

$$\omega^2\left(k - \omega^2 m\right)\left(\omega^2 mM - 2km - kM\right) = 0 \tag{9.63}$$

The resonant or normal frequencies of the system are

$$\omega_1 = 0 \qquad \omega_2 = \left(\frac{k}{m}\right)^{1/2} \qquad \omega_3 = \left(\frac{k}{m} + \frac{2k}{M}\right)^{1/2} \tag{9.64}$$

Normal Modes To find the normal modes of the system, the

expanded form of Eq. (9.14) is needed. To each value of *w* we will have a set of *a*'s. To distinguish them, we shall add an additional subscript *p*.

$$(k - \omega_p^2 m)\, a_{1p} - k a_{2p} + 0 = 0 \qquad (9.65a)$$

$$-k a_{1p} + (2k - \omega_p^2 M)\, a_{2p} - k a_{3p} = 0 \qquad (9.65b)$$

$$0 - k a_{2p} + (k - \omega_p^2 m)\, a_{3p} = 0 \qquad (9.65c)$$

Substituting the frequency $w_1 = 0$, from Eqs. (9.65a) and (9.65c) $a_{11} = a_{21}$ and $a_{21} = a_{31}$.

Hence,

$a_{11} = a_{21} = a_{31} = a$ (9.66) That is, the displacements of all the atoms are equal and are in the same direction. It clearly shows that this mode is not an oscillation but a pure translation of the system as a whole and is illustrated in Fig. 9.5 (a).

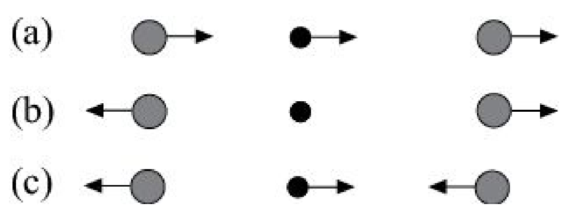


Fig. 9.5 Longitudinal modes of vibration of a CO_2 molecule: (a) translational mode; (b) symmetric stretching mode; (c) antisymmetric streching mode.

Setting $\omega = \omega_2 = \sqrt{k/m}$, from Eq. (9.65a) $a_{22} = 0$ and from Eq. (9.65b) $a_{12} = -\, a_{32}$. Hence, $a_{12} = -\, a_{32} = b\ a_{22} = 0$ (9.67) In this mode, the centre atom is at rest while the outer ones oscillate exactly out of phase, which is illustrated in Fig. 9.5 (b). This mode is called a **symmetric stretching mode** since both the bonds either stretch or compress at the same time.

Setting $\omega = \omega_3 = \left(\frac{k}{m} + \frac{2k}{M}\right)^{1/2}$ in Eqs. (9.65a) and (9.65c), we get

$$a_{13} = -(M/2m)a_{23} \quad \text{and} \quad a_{33} = -(M/2m)a_{23}$$

Hence,

$$a_{13} = a_{33} = \gamma \qquad \text{and} \qquad a_{23} = \frac{-2m}{M}\gamma \tag{9.68}$$

Hence, the two outer atoms vibrate with the same amplitude, while the inner one oscillates out of phase with them with a different amplitude. This mode of vibration is illustrated in Fig. 9.5 (c), which is known as **asymmetric** or **antisymmetric stretching mode,** since when one bond gets compressed the other gets elongated.

Normal Coordinates From Eqs. (9.66), (9.67) and (9.68) the matrix of eigenvectors *A* can be written as

$$A=\begin{pmatrix}\alpha & \beta & \gamma\\ \alpha & 0 & \dfrac{-2m\gamma}{M}\\ \alpha & -\beta & \gamma\end{pmatrix} \tag{9.69}$$

The condition given in Eq. (9.21), $\tilde{A}GA=1$ gives

$$\begin{pmatrix}\alpha & \alpha & \alpha\\ \beta & 0 & -\beta\\ \gamma & \dfrac{-2m\gamma}{M} & \gamma\end{pmatrix}\begin{pmatrix}m & 0 & 0\\ 0 & M & 0\\ 0 & 0 & M\end{pmatrix}\begin{pmatrix}\alpha & \beta & \gamma\\ \alpha & 0 & \dfrac{-2m\gamma}{M}\\ \alpha & -\beta & \gamma\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{9.70}$$

Simplification gives

$$\alpha=\frac{1}{(2m+M)^{1/2}} \qquad \beta=\frac{1}{(2m)^{1/2}} \qquad \gamma=\left[\frac{M}{2m(M+2m)}\right]^{1/2} \tag{9.71}$$

The normal co-ordinates Q_1, Q_2 and Q_3 of the system are given by

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}\alpha & \beta & \gamma\\ \alpha & 0 & \dfrac{-2m\gamma}{M}\\ \alpha & -\beta & \gamma\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\\ Q_3\end{pmatrix} \tag{9.72}$$

where *a*, *b* and *g* are given by Eq. (9.71). On simplification

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} Q_1\alpha & + & Q_2\beta + Q_3\gamma \\ Q_1\alpha & - & (2m/M)Q_3\gamma \\ Q_1\alpha & - & Q_2\beta + Q_3\gamma \end{pmatrix}$$

Solving the resulting 3 equations for Q_1, Q_2 and Q_3

$$Q_1 = \frac{mx_1 + Mx_2 + mx_3}{(M + 2m)\alpha} = \frac{mx_1 + Mx_2 + mx_3}{(M + 2m)^{1/2}} \tag{9.73a}$$

$$Q_2 = \frac{x_1 - x_3}{2\beta} = \frac{x_1 - x_3}{(2/m)^{1/2}} \tag{9.73b}$$

$$Q_3 = \frac{M\left(x_1 - 2x_2 + x_3\right)}{(2M + 4m)\gamma} = \frac{m^{1/2}M^{1/2}(x_1 - 2x_2 + x_3)}{(2M + 4m)^{1/2}} \tag{9.73c}$$

The vibrations discussed so far are the longitudinal ones. In the molecule, there will also be normal modes of vibrations perpendicular to the axis. It is evident from Eq. (9.64) that $w_3 > w_2$. In general, the mode that has higher symmetry will have the lower frequency. The antisymmetric mode has lower symmetry and therefore it has a higher frequency.

WORKED EXAMPLES

Example 9.1 A simple pendulum has a bob of mass m with a mass m_1 at the moving support (pendulum with moving support). Mass m_1 moves on a horizontal line in the vertical plane in which the pendulum oscillates. Find the normal frequencies and normal modes of vibrations.

Solution: From Example 3.8 and Fig. 3.5

$$T = \frac{1}{2}(m_1 + m)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta)$$

where m_1 is the mass at the support and m is the mass of the pendulum bob and l its length. The coordinates be x and q. Since cos$q$ = 1 at the equilibrium position, the G matrix is given by $G = \begin{pmatrix} m_1 + m & ml \\ ml & ml^2 \end{pmatrix}$

Taking the point of support as the zero of potential energy V

$V_{11} = 0$ $V_{12} = V_{21} = 0$ $V_{22} = mgl$ The secular determinant is

$$\begin{vmatrix} -\omega^2(m_1 + m) & -ml\omega^2 \\ -ml\omega^2 & mgl - \omega^2 ml^2 \end{vmatrix} = 0$$

$$-\omega^2(m_1 + m)(mgl - \omega^2 ml^2) - m^2 l^2 \omega^4 = 0$$

$$\omega^2\left(\omega^2 - \frac{m_1 + m}{m_1}\frac{g}{l}\right) = 0$$

$$\omega_1^2 = 0 \qquad \text{and} \qquad \omega_2^2 = \frac{m_1 + m}{m_1}\frac{g}{l}$$

The normal frequencies are $\omega_1 = 0$ and $\omega_2 = \sqrt{\left(\frac{m_1 + m}{m_1}\right)\frac{g}{l}}$, where $\omega_1 = 0$ corresponds to translation of the two masses. Using Eq. (9.14)

$$-\omega^2 (m_1 + m) a_{12} - ml\omega^2 a_{22} = 0$$

Substituting the value of ω_2

$$a_{12} = \frac{-ml}{m_1 + m} a_{22}$$

That is, the masses will be moving in the opposite direction during the oscillation. The two normal modes are presented in Fig. 9.6.

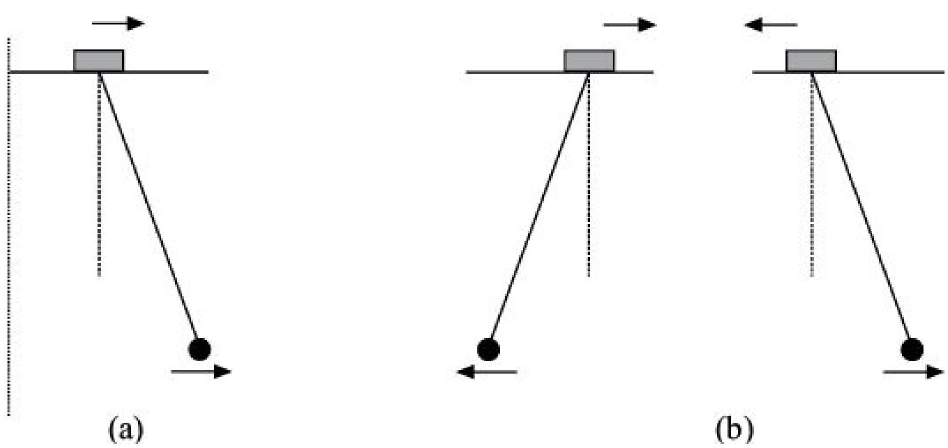


Fig. 9.6 The normal modes of a pendulum with moving support. (a) Translational mode, (b) Represents the frequency w_2.

Example 9.2 Find the normal frequencies and normal modes for a double pendulum, each having a mass *m* suspended by a string of length *l*.

Solution: Fig. 9.7 illustrates a double pendulum in a displaced position.

$$x_1 = l\sin\theta \qquad y_1 = l\cos\theta$$

$$x_2 = l\sin\theta + l\sin\phi \qquad y_2 = l\cos\theta + l\cos\phi$$

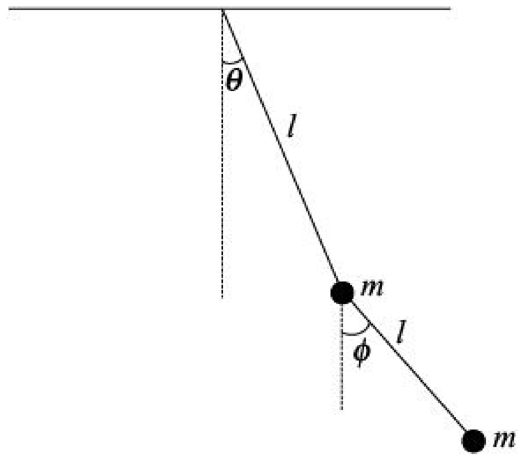


Fig. 9.7 Double pendulum.

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$T = ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + ml^2\cos(\theta - \phi)\dot{\theta}\dot{\phi}$$

The reference level for potential energy is at distance $2l$ below the point of suspension.

The height of the upper mass above the zero level = $l + l\,(1 - \cos\theta)$

The height of the lower mass above zero level = $l\,(1 - \cos\theta) + l\,(1 - \cos\phi)$

$$V = 4mgl - 2mgl\cos\theta - mgl\cos\phi$$

$$V_{11} = \left(\frac{\partial^2 V}{\partial\theta^2}\right)_{\substack{\theta=0\\ \phi=0}} = 2mgl \qquad V_{12} = \left(\frac{\partial^2 V}{\partial\theta\partial\phi}\right)_{\substack{\theta=0\\ \phi=0}} = 0$$

$$V_{22} = \left(\frac{\partial^2 V}{\partial\phi^2}\right)_{\substack{\theta=0\\ \phi=0}} = mgl \qquad V_{21} = 0$$

The elements of the G-matrix are

$$G_{11} = 2ml^2 \qquad G_{22} = ml^2 \qquad G_{12} = G_{21} = ml^2$$

The normal frequencies are given by

$$\begin{vmatrix} 2mgl - \omega^2 2ml^2 & -\omega^2 ml^2 \\ -\omega^2 ml^2 & mgl - \omega^2 ml^2 \end{vmatrix} = 0$$

$$2(mgl - \omega^2 ml^2)\,(mgl - \omega^2 ml^2) - \omega^4 m^2 l^4 = 0$$

$$2\left(\frac{g}{l} - \omega^2\right)\left(\frac{g}{l} - \omega^2\right) - \omega^4 = 0$$

$$\left[\sqrt{2}\left(\frac{g}{l} - \omega^2\right) + \omega^2\right]\left[\sqrt{2}\left(\frac{g}{l} - \omega^2\right) - \omega^2\right] = 0$$

$$\omega^2 = \frac{\sqrt{2}\, g/l}{\sqrt{2}+1} \qquad \text{or} \qquad \frac{\sqrt{2}\, g/l}{\sqrt{2}-1}$$

$$\omega_1^2 = \frac{\sqrt{2}\left(\sqrt{2}-1\right) g/l}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} = \left(2-\sqrt{2}\right)\frac{g}{l}$$

$$\omega_2^2 = \frac{\sqrt{2}\left(\sqrt{2}+1\right) g/l}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)} = \left(2+\sqrt{2}\right)\frac{g}{l}$$

The normal frequencies are

$$\omega_1 = \left[\left(2-\sqrt{2}\right)\frac{g}{l}\right]^{1/2} \quad \text{and} \quad \omega_2 = \left[\left(2+\sqrt{2}\right)\frac{g}{l}\right]^{1/2}$$

Using Eq. (9.14), for ω_1 we get the following two equations:

$$\left[2mgl - \left(2-\sqrt{2}\right)\frac{g}{l}2ml^2\right]a_{11} - \left(2-\sqrt{2}\right)\frac{g}{l}ml^2 a_{21} = 0$$

$$-\left(2-\sqrt{2}\right)\frac{g}{l}ml^2\, a_{11} + \left[mgl - \left(2-\sqrt{2}\right)\frac{g}{l}ml^2\right]a_{21} = 0$$

Simplifying

$$\left(2\sqrt{2}-2\right)a_{11} - \left(2-\sqrt{2}\right)a_{21} = 0$$

$$-\left(2-\sqrt{2}\right)a_{11} - \left(1-\sqrt{2}\right)a_{21} = 0$$

From these, $a_{11} = 1, a_{21} = \sqrt{2},$ the two displacements are in the same direction. Again using Eq. (9.14) for ω_2, the following two equations are obtained:

$$\left[2mgl - \left(2+\sqrt{2}\right)\frac{g}{l}2ml^2\right]a_{12} - \left(2+\sqrt{2}\right)\frac{g}{l}ml^2 a_{22} = 0$$

$$-\left(2+\sqrt{2}\right)\frac{g}{l}ml^2 a_{12} + \left[mgl - \left(2+\sqrt{2}\right)\frac{g}{l}ml^2\right]a_{22} = 0$$

From these we have $a_{12} = 1$ and $a_{21} = -\sqrt{2},$ and the two displacements are in opposite direction. The matrix of eigenvectors is

$$\begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

Consequently,

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} Q_1 + Q_2 \\ \sqrt{2}\,Q_1 - \sqrt{2}\,Q_2 \end{pmatrix}$$

$$Q_1 = \frac{\theta}{2} + \frac{\phi}{2\sqrt{2}} \quad \text{and} \quad Q_2 = \frac{\theta}{2} - \frac{\phi}{2\sqrt{2}}$$

To study the nature of a particular mode, say Q_1, set the other normal co-ordinate to zero.

$$Q_2 = 0 \qquad \text{or} \qquad \frac{\theta}{2} - \frac{\phi}{2\sqrt{2}} = 0 \qquad \text{or} \qquad \theta = \frac{\phi}{\sqrt{2}}$$

That is, in the Q_1 mode, both the masses have displacements in the same direction as shown in Fig. 9.8(a). This is called the *symmetric mode*. The Q_2 mode appears when $Q_1 = 0$ or $\theta = -\phi/\sqrt{2}$. That is, Q_2 mode corresponds to displacement of the masses in the opposite direction as shown in Fig. 9.8(b). This mode is known as *antisymmetric mode*.

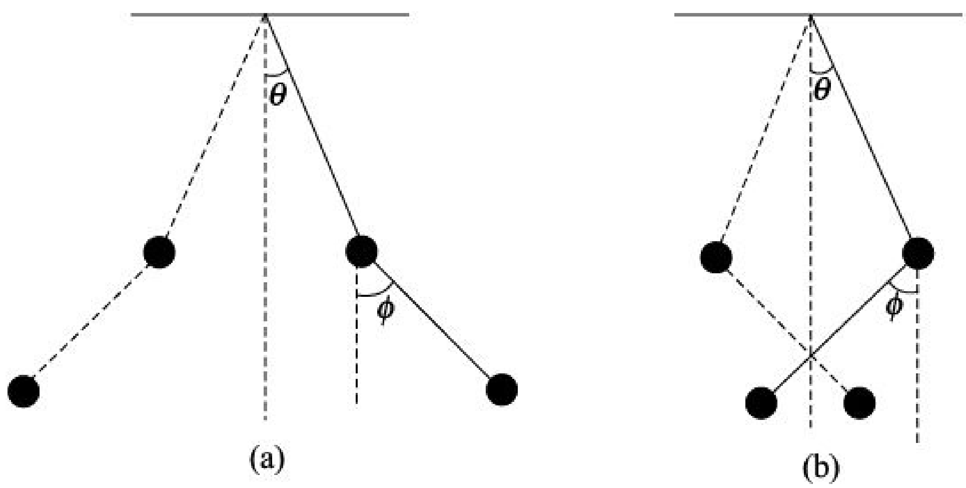


Fig. 9.8 Normal vibrations of a double pendulum: (a) symmetric mode; (b) antisymmetric mode.

Example 9.3 Consider a system of two harmonic oscillators coupled by a spring of spring constant k_1. The spring constant of the harmonic oscillators is k and the mass connected to each of the oscillator is m. Find the normal frequencies and the normal coordinates of the system.

Solution: The system has two degrees of freedom represented by the displacements x_1 and x_2 shown in Fig. 9.9.

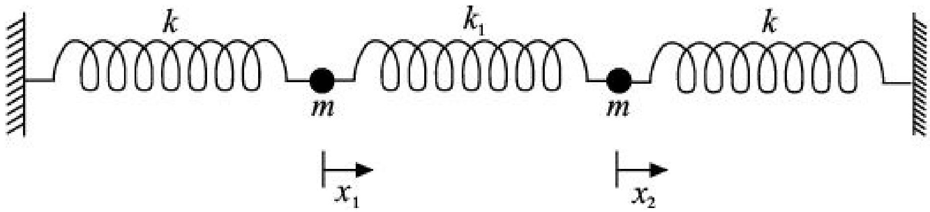


Fig. 9.9 Two harmonic oscillators coupled by a spring.

$$\text{K.E } T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$\text{P.E } V = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k_1(x_1 - x_2)^2 = \frac{1}{2}(k + k_1)x_1^2 + \frac{1}{2}(k + k_1)x_2^2 - k_1x_1x_2$$

$$V_{11} = k + k_1 \qquad V_{22} = k + k_1 \qquad V_{12} = -k_1$$

$$G_{11} = m \qquad G_{22} = m \qquad G_{12} = G_{21} = 0$$

The normal frequencies can be obtained from the secular determinant

$$\begin{vmatrix} k + k_1 - \omega^2 m & -k_1 \\ -k_1 & k + k_1 - \omega^2 m \end{vmatrix} = 0 \tag{v}$$

$$\left(k + k_1 - \omega^2 m\right)^2 - k_1^2 = 0$$

$$(k + 2k_1 - \omega^2 m) = 0 \qquad \text{or} \qquad (k - \omega^2 m) = 0$$

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{k}{m} + \frac{2k_1}{m}}$$

Frequency ω_1 is independent of k_1. It implies that the spring coupling the two oscillators is not participating in this mode of oscillation. From Eq. (9.14)

$$(k + k_1 - \omega_p^2 m)a_{1p} - k_1 a_{2p} = 0$$

$$-k_1 a_{1p} + (k + k_1 - \omega_p^2 m)a_{2p} = 0$$

Substituting the frequency $\omega_p = \omega_1 = \sqrt{k/m}$

$$k_1(a_{11} - a_{21}) = 0 \qquad \text{or} \qquad a_{11} = a_{21} = \alpha$$

That is, the displacements are equal and in phase and therefore no change to the spring that couples the two oscillators. This is understandable as its spring constant is not appearing in the frequency expression. The mode of oscillation is illustrated in Fig. 9.10 (a).

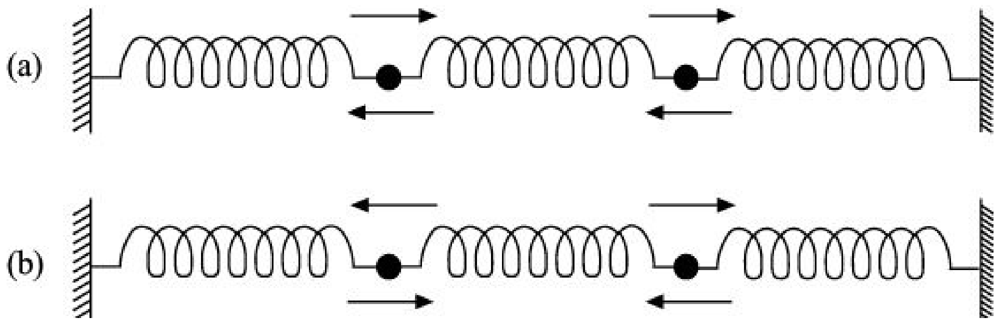


Fig. 9.10 Modes of vibration of two coupled oscillators.

Substituting the frequency
$$\omega_p = \omega_2 = \sqrt{\frac{k}{m} + \frac{2k_1}{m}}$$

$$-k_1 (a_{12} + a_{21}) = 0 \qquad \text{or} \qquad a_{12} = -a_{21} = \beta$$

The displacements are equal but in opposite directions. This mode of oscillation is illustrated in Fig. 9.10 (b). As expected, the spring that couples the two oscillators gets compressed and elongated alternately. The matrix of eigenvectors

$$A = \begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix}$$

The condition Eq. (9.21) gives

$$\begin{pmatrix} \alpha & \alpha \\ \beta & -\beta \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2m\alpha^2 & 0 \\ 0 & 2m\beta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$2m\alpha^2 = 1 \qquad \alpha^2 = \frac{1}{2m} \quad \text{or} \quad \alpha = \sqrt{\frac{1}{2m}}$$

$$2m\beta^2 = 1 \qquad \beta = \sqrt{\frac{1}{2m}}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & \frac{-1}{\sqrt{2m}} \end{pmatrix}$$

From Eq. (9.23)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & -\frac{1}{\sqrt{2m}} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

$$x_1 = \frac{1}{\sqrt{2m}}(Q_1 + Q_2) \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2m}}(Q_1 - Q_2)$$

Solving

$$Q_1 = \sqrt{\frac{m}{2}}(\theta_1 + \theta_2) \quad \text{and} \quad Q_2 = \sqrt{\frac{m}{2}}(\theta_1 - \theta_2)$$

The nature of any one of the normal modes can be investigated by setting the other normal modes to zero. In this way also, we can get the different modes of vibrations of the system.

REVIEW QUESTIONS

1. Explain stable, unstable and neutral equilibria on the basis of potential function.
2. For small displacements, the condition for stable equilibrium is that the potential energy is minimum at the equilibrium configuration. Substantiate.
3. Outline the procedure for obtaining the elements of the V and G matrices of a system.
4. Explain (i) normal modes of vibration; (ii) normal coordinates; and (iii) normal frequencies of a system.
5. Express the kinetic and potential energies of a system in terms of normal coordinates.
6. Sketch the normal modes of vibration of a CO_2 molecule in the increasing order of frequency.

PROBLEMS

1. Consider a diatomic molecule consisting of masses m_1 and m_2 connected by a spring of spring constant k vibrating along the line joining the two masses. Obtain its normal frequencies and normal modes of vibration.
2. A system of two harmonic osicllators having spring constant k is coupled by a spring of spring constant k. If the mass connected to each of the harmonic oscillators is m, show that the system has the normal frequencies $\sqrt{\frac{k}{m}}$ and $\sqrt{\frac{3k}{m}}$.
3. A spring of force constant k hanging from a rigid support carries a mass m at the other end. An identical spring carrying a mass m is connected to the first mass. The system is allowed to oscillate in the vertical direction. Find the frequencies of the normal modes of vibration. Also find the ratios of the amplitudes of the two masses in the two modes.
4. Obtain the two resonant frequencies for the double pendulum assuming unequal masses and lengths. Discuss the following cases: (i) $m_1 > m_2$; (ii) $m_1 < m_2$; (iii) $m_1 = m_2 = m$ and $l_1 = l_2 = l$.
5. The masses of the bobs of two pendulums are m_1 and m_2. The bobs are coupled by a spring of force constant k. If their lengths are equal to l, obtain the normal frequencies of the system.
6. Three equal mass points m are connected by 4 springs of same force constant

k as shown in Fig. 9.11. The point O is fixed. When the system is set into vibrations, the mass points and the springs are constrained to move only on a circle. Determine the resonant frequencies.

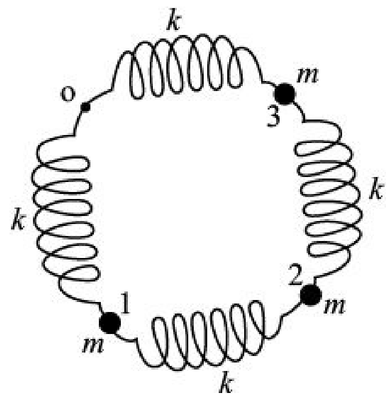


Fig. 9.11 Arrangement of springs and masses on the circumference of a circle.

10

Special Theory of Relativity

The theories developed during three centuries, starting from 1600 AD, had been very successful in explaining most of the phenomena in physical science. Newtonian mechanics explained the dynamics of objects on earth and in the heavens. It successfully explained wave motion and the behaviour of fluids. The kinetic theory of matter showed the connection between mechanics and heat. Maxwell's electromagnetic theory unified the branches of optics, electricity and magnetism into a single larger field called **electrodynamics.** Towards the end of the 19th century, certain new discoveries (X-rays, radioactivity and electron) and experimental observations (blackbody radiation curves, photoelectric effect, optical spectra, etc.) were made, which the existing theories failed to explain. Yet there was a sense of completion among the physicists that they would be able to explain these phenomena on the basis of existing theories. However, with the formulation of two revolutionary new theories, quantum theory and theory of relativity, they were convinced about the inability of classical physics to explain all the physical phenomena. We discuss the special theory of relativity, which was proposed by Albert Einstein in 1905, in this chapter.

10.1 GALILEAN TRANSFORMATION

Newton's first law or the law of inertia states that a system at rest will remain at rest or a system in uniform motion will remain in uniform motion if no net external force acts on it. Systems in which the law of inertia holds are called **inertial systems**. A reference frame that moves with uniform velocity with respect to an inertial frame is also an inertial frame. For most purposes, a set of coordinate axes attached to the earth may be regarded as an inertial frame of reference. Here, the small acceleration resulting from the rotational motion about its own axis and the orbital motion about the sun is neglected. An ideal inertial frame is a coordinate frame of reference fixed in space with respect to fixed stars. Accelerating frames of reference are **noninertial frames**.

Consider two inertial systems *S* and *S* with coordinate axes *xyz* and *xyz* attached to them. Let *S* be moving with respect to *S* with a uniform velocity **v** along the *xx* axes as shown in Fig. 10.1. The origins of the two systems coincide when *t* = *t* = 0. Let an event be taking place at *P* whose coordinates with respect to *S* be (*x*, *y*, *z*, t) and with respect to *S* be (*x*, *y*, z, *t*) From Fig. 10.1 it is obvious that these coordinates are related by

$$x' = x - \mathrm{v}t$$

$$y' = y \qquad (10.1)$$

$$z' = z$$

$$t' = t$$

These are called the **Galilean transformation equations or Newtonian transformation equations.**

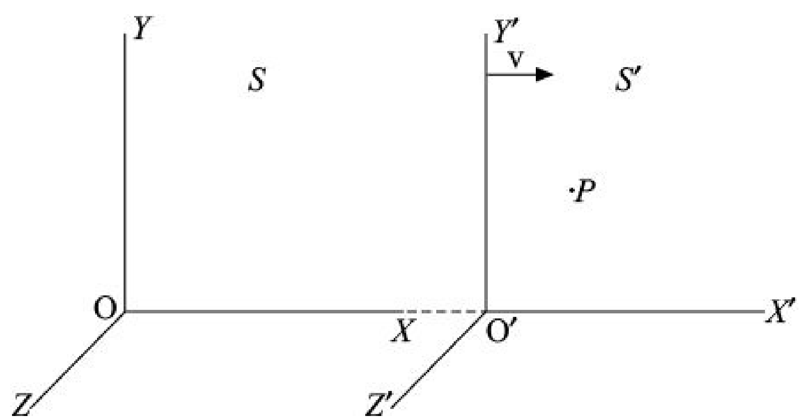


Fig. 10.1 The inertial system *S* and *S* with coordinates axes *xyz* and *xyz*.

The **Galilean velocity transformation** is obtained by differentiating the above equations with respect to time and using the result (*d*/*dt*) = (*d*/*dt*):

$$\frac{dx'}{dt'} = \frac{dx}{dt} - \text{v} \qquad u'_x = u_x - \text{v}$$

$$\frac{dy'}{dt'} = \frac{dy}{dt} \quad \text{or} \quad u'_y = u_y \qquad (10.2)$$

$$\frac{dz'}{dt'} = \frac{dz}{dt} \qquad u'_z = u_z$$

We can get the acceleration transformation equation by taking the derivative of Eq. (10.2) with respect to time.

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} \qquad a'_x = a_x$$

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2} \quad \text{or} \quad a'_y = a_y \qquad (10.3)$$

$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2} \qquad a'_z = a_z$$

That is, acceleration is invariant with respect to Galilean transformation. In Newtonian formulation, mass is absolute. Multiplying Eq. (10.3) by *m*, we have

$$ma'_x = ma_x$$

$$ma'_y = ma_y \quad \text{or} \quad \mathbf{F}' = \mathbf{F} \qquad (10.4)$$

$$ma'_z = ma_z$$

This implies that the force on a particle of mass *m* at the point *P* is identical in the two inertial frames. That is, Newton's second law is invariant under Galilean transformation. It can also be shown that the other laws of mechanics also satisfy this principle. This result means that the basic laws of physics are the same in all inertial reference frames, which is the principle of Galilean–Newtonian relativity. In other words, no inertial frame is special and all inertial frames are equivalent.

Newtonian relativity assumes that space and time are absolute quantities. Their measurement does not change from one inertial frame to another. The mass of an object and force are unchanged by a change in inertial frame. But the position of an object and its velocity are different in different inertial frames.

10.2 ELECTROMAGNETISM AND GALILEAN TRANSFORMATION

Maxwell's equations predicted the existence of electromagnetic waves propagating through space with a speed of 3 10^8 m/s. Then, a spherical electromagnetic wave propagating with a constant speed ***c*** in the reference frame *S* is given by

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad (10.5)$$

For this equation to be invariant, its form in the system S' should be

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \quad (10.6)$$

Substituting the values of x, y, z, t in terms of x', y', z', t', we get

$$(x' + \mathrm{v}t)^2 + y'^2 + z'^2 - c^2t'^2 = 0 \quad (10.7)$$

which is not the same as Eq. (10.6). Hence, the Galilean transformation equations do not hold good in the case of electromagnetism. This seemed to suggest that there must be some special reference frame wherein the velocity of light is
3 10^8 m/s.

The 19th century physicists used to view the various phenomena in terms of the laws of mechanics. The mechanical wave phenomena require a medium to support the wave. Therefore, it was natural for the physicists to assume that light and other electromagnetic waves too must travel in some medium. They called this transparent medium *ether* and assumed that it permeates all space. They had to assign very strange properties to ether. It had to be transparent and massless so that electromagnetic waves could travel through vacuum. On the contrary, it had to be very hard to support transverse vibrations of the wave motion. The ether hypothesis led to the following two alternatives: (i) **The stationary ether hypothesis** wherein the ether is at rest with respect to the bodies moving through it. The reference frame wherein the ether could be considered at rest is called the **ether frame** (or the **rest frame** or **absolute frame)**. In this frame the velocity of light is always **c**.

(ii) **The ether drag hypothesis** wherein ether is dragged along with the bodies which move through it.

A number of experiments were designed to check the ether hypothesis. Of these, the most direct one is the one performed by Michelson and Morley in the 1880s.

10.3 MICHELSON–MORLEY EXPERIMENT

The purpose of the Michelson-Morley experiment was to confirm the existence of an absolute frame of reference (stationary ether). If the ether is at rest, when the earth moves through it there must be a relative velocity of the earth with respect to the ether. What they did was to measure the difference in the speed of light in different directions.

The Interferometer

The experimental set-up used is the Michelson interferometer shown in Fig. 10.2. The light from a source is split into two beams by a half-silvered plate P. One beam travels to mirror M_1 and the other to mirror M_2. The beams are reflected by M_1 and M_2 and are recombined again after passing through P. Beam 2 goes through the plate P three times, whereas beam 1 goes through P only once. Hence, to make the optical paths of the two beams equal, a compensating plate P is placed in the path of beam 1. Beams 1 and 2 arrive at the telescope T and produce interference fringes. If the optical path lengths of the beams are exactly equal, constructive interference occurs, leading to a bright fringe. If one mirror is moved a distance $l/4$ which corresponds to a path difference of l/2 between the beams, destructive interference occurs, giving rise to a dark fringe. Thus, by moving one of the mirrors, the fringe system can be made to move past a crosswire which serves as the reference mark. Let the earth be moving to the right with a velocity **v** with respect to the stationary ether. (See Fig.10.2.) Michelson arranged the interferometer in such a way that PM_1 is parallel to the direction of the vector **v**. To reduce mechanical vibrations, the interferometer was mounted on a large stone that floated in a tank of mercury.

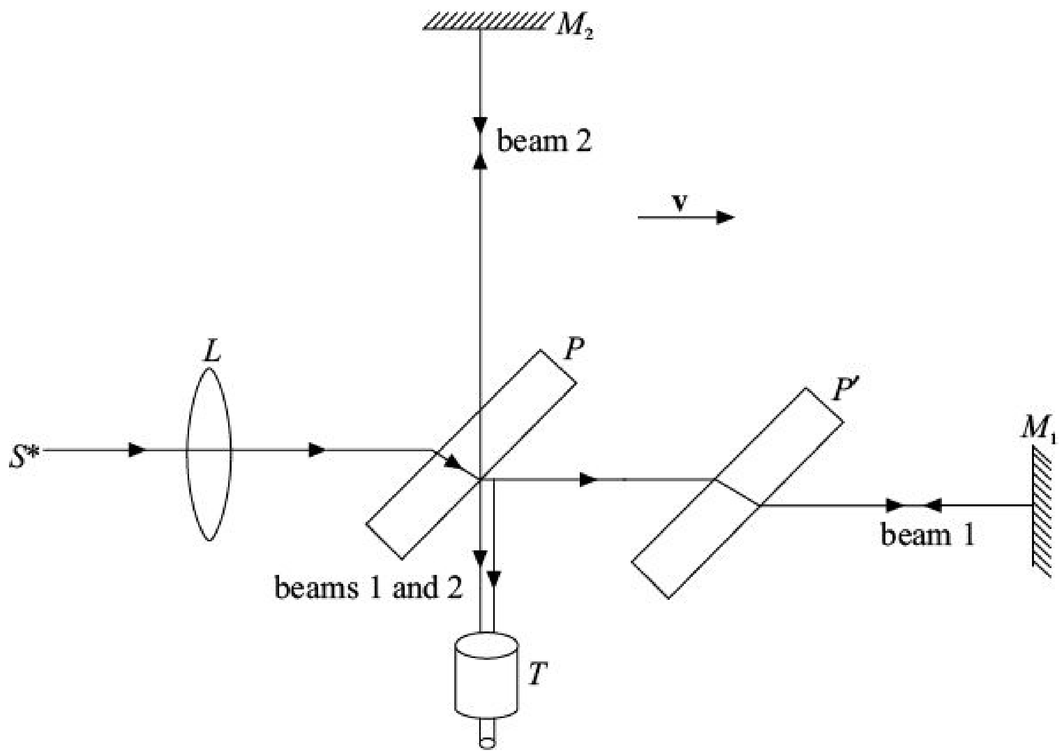


Fig. 10.2 Schematic representation of the Michelson-Morley experiment.

The Experiment

To start with, the mirrors M_1 and M_2 are set such that $PM_1 = PM_2 = d$. If the apparatus is stationary in ether, the two waves take the same time to return to the telescope and hence meet in the same phase. But the apparatus is moving with the same velocity **v** to the right. Therefore, the time required by the two waves for their to and fro journeys through the same distance will not be equal. First, we consider beam 1 which travels parallel to the velocity **v**. The transmitted wave travels towards M_1 with relative velocity c – **v**. After reflection at M_1 it travels towards the glass plate P with relative velocity c + v. Hence, the time required by this wave for its round trip is

$$t_1 = \frac{d}{c - \mathrm{v}} + \frac{d}{c + \mathrm{v}} = \frac{2cd}{c^2 - \mathrm{v}^2} = \frac{2d}{c(1 - \mathrm{v}^2 / c^2)} \qquad (10.8)$$

The path of beam 2 when the interferometer is moving with velocity **v** parallel to beam 1 is illustrated in Fig.10.3 (a). By vector addition, the velocity component perpendicular to the direction of motion of the interferometer is $(c^2 - \mathbf{v}^2)^{1/2}$ (See Fig. 10.3b). The time taken by beam 2 to travel from P to M_2 and back is

$$t_2 = \frac{2d}{(c^2 - v^2)^{1/2}} = \frac{2d}{c(1 - v^2/c^2)^{1/2}} \tag{10.9}$$

The difference between the time taken by the two beams is

$$\Delta t = t_1 - t_2 = \frac{2d}{c}\left[\left(1 - \frac{v^2}{c^2}\right)^{-1} - \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right]$$

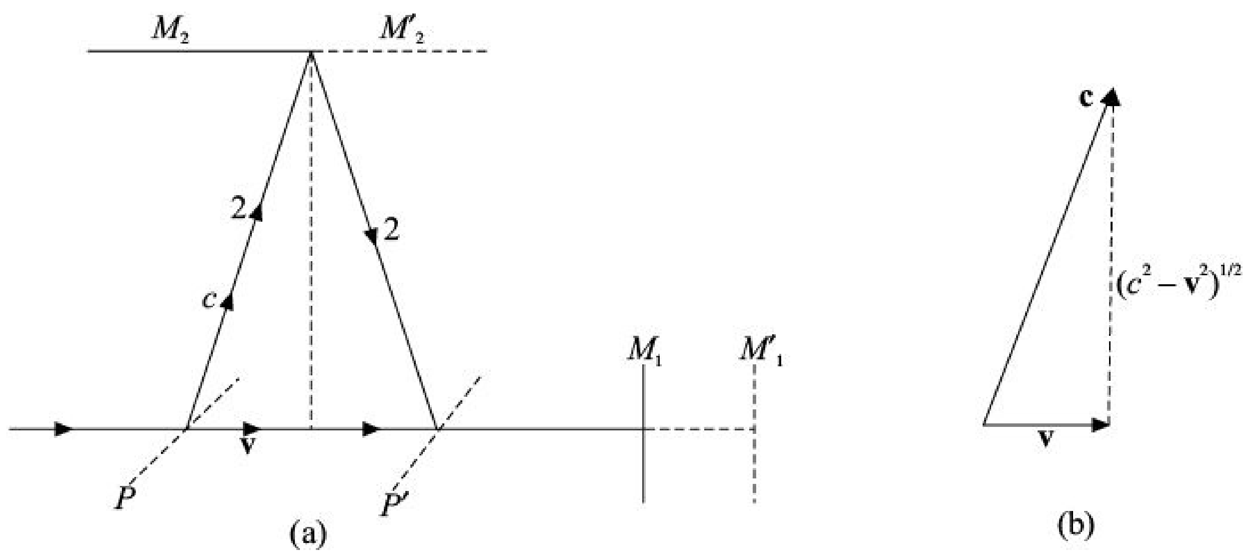


Fig. 10.3 (a) The path of beam 2 while the interferometer is moving with velocity **v** parallel to PM_1; (b) vector addition of the velocities **v** and ***c***.

Since v/*c* << 1, using binomial expansion

$$\Delta N = \frac{\text{Path difference}}{\text{Wavelength}} = \frac{2d\,v^2/c^2}{\lambda} = \frac{2dv^2}{c^2\lambda} \tag{10.10}$$

If **v** = 0, then *Dt* = 0 and the two beams take the same time.

In their experiment, Michelson and Morley rotated the interferometer through an angle of 90°. In the rotated position, beam 2 will be parallel to the velocity **v** and beam 1 perpendicular to it. This leads to a total difference in time (*Dt*) + (*Dt*) which is equivalent to a path difference of 2 (*Dt*)**c** = $2dv^2/c^2$. Therefore, there would be a shift in the fringe system across the crosswire in the telescope. The number of fringes shifted

$$\Delta N = \frac{\text{Path difference}}{\text{Wavelength}} = \frac{2d\ v^2 / c^2}{\lambda} = \frac{2dv^2}{c^2\lambda} \qquad (10.11)$$

Michelson and Morley reflected beams 1 and 2 back and forth 8 times so that the path length is 10m. With $\lambda = 5000$ Å, Eq. (10.11) leads to a fringe shift of

$$\Delta N = \frac{2(10\text{m})\,(3\times 10^4\,\text{ms}^{-1})^2}{(3\times 10^8\,\text{ms}^{-1})^2\,(5\times 10^{-7}\,\text{m})} = 0.4 \qquad (10.12)$$

Detection of this fringe width is possible, since their apparatus was capable of observing a fringe shift as small as 0.01 fringe. But they could observe no significant fringe shift. They repeated the experiment at different places at different times of the day and at different seasons. But they did not observe a significant fringe shift, indicating the absence of ether and that the speed of light in the interferometer is the same for the two perpendicular paths. This null result was one of the great puzzles of physics towards the end of the 19th century.

The null result can be explained if the Galilean transformation is abandoned and the velocity of light is assumed to be the same in all inertial frames. Then for beam 1, $t_1 = 2d/\mathbf{c}$ and for beam 2, $t_2 = 2d/\mathbf{c}$, which leads to $Dt = t_1 - t_2 = 0$. It is also evident from Eq. (10.11) that DN approaches zero, if $\mathbf{v} << \mathbf{c}$. Hence, we may assume that the Galilean transformation is valid when $\mathbf{v} << \mathbf{c}$. In other words, the Galilean transformation is valid for mechanics but not for electromagnetism, since the velocity of electromagnetic waves is equal to **c**.

The results of Michelson and Morley were a real challenge and a number of explanations were put forth over a period of years. The radical new theory proposed by Einstein in 1905 explained the various experimental results satisfactorily and changed our ideas about space and time.

10.4 THE POSTULATES OF SPECIAL THEORY OF RELATIVITY

Einstein in his theory dropped the concept of ether and the accompanying assumption of an absolute frame of reference at rest. Also, he revised the classical ideas regarding space and time by asserting that absolute motion is meaningless. Einstein's ideas are embodied in two postulates. The first one is an extension of the Newtonian principle of relativity to include not only the laws of mechanics but also those of the rest of physics.

Postulate 1–The principle of equivalence: The laws of physics have the same form in all inertial reference frames.

Postulate 2–Constancy of the speed of light: The speed of light in free space (vacuum) is always a constant **c** and is independent of the speed of the source, the observer or the relative motion of the inertial systems.

These two postulates form the foundation of Einstein's special theory of relativity. It is referred to as *special* to distinguish it from his later theory, the general theory of relativity, which deals with noninertial frames.

10.5 LORENTZ TRANSFORMATION

The transformation equations for inertial frames of reference moving with uniform relative velocity were derived by Einstein. However, they are called *Lorentz transformations* since Lorentz derived (in 1890) the same relations in electromagnetism. Consider two reference frames S and S moving with uniform relative motion as described in Section 10.1. Let two observers O and O observe any event P from systems S and S, respectively. Let the event P is produced at t = 0 when the origins of the two frames coincide. For the observer at O, the coordinates of the event at a particular instant be (x, y, z, t). The same event is described by the coordinates (x, y, z, t) for the observer O on the system S.

The velocity of S with respect to S is along the x-axis. Hence, $y = y$ and $z = z$ (10.13) According to postulate 1, a uniform rectilinear motion in S must go over into a uniform rectilinear motion in S. Hence, the transformation relating x and x must be linear. A non-linear transformation may produce acceleration in S even if the velocity is constant in S. In addition, the transformation must reduce to Galilean transformation at low speeds. Therefore, the transformation equation relating
x and x can be written as $x = k(x - \text{v}t)$ (10.14) where k is independent of x and t. Since S is moving relative to S with velocity **v** along the positive x-axis, $x = k(x + \text{v}t)$ (10.15) The same constant k is used, since according to the first postulate nothing distinguishes S and S from one another except the sign of the relative velocity. Substituting the value of x from Eq. (10.14), we have

$$x = k\,[k(x - vt) + vt']$$

or,
$$t' = kt + \frac{(1 - k^2)}{kv}\,x \qquad (10.16)$$

We then write the tentative form of the transformation equations as:

$$x' = k(x - vt)$$
$$y' = y$$
$$z' = z \qquad (10.17)$$
$$t' = kt + \frac{(1 - k^2)x}{kv}$$

Our next step is to get the explicit form of the constant *k*. Let us assume that the event is a pulse of light emitted from O at time $t = 0$. The pulse of light spreads as a spherical wave travelling with the velocity *c*. The equation of the wavefront in *S* at time *t* is: $x^2 + y^2 + z^2 = c^2\,t^2$ (10.18) The wavefront in the reference frame *S* is $x'^2 + y'^2 + z'^2 = c^2\,t'^2$ (10.19)

Here, we have used the second postulate that the velocity of the light wave is the same in all directions in either frame of reference. Substituting the tentative transformation equations, Eq. (10.17) in Eq. (10.19), we have

....

$$k^2\,(x - vt)^2 + y^2 + z^2 = c^2k^2t^2 + \frac{2c^2tx\,(1 - k^2)}{v} + \frac{c^2x^2\,(1 - k^2)^2}{k^2v^2} \qquad (10.20)$$

We must choose *k* such that Eq. (10.20) reduces to Eq. (10.18), since each equation represents the position of the wavefront as measured in *S*. Comparing the coefficients of the terms in *x* in both equations

$$2k^2\text{v}t + \frac{2c^2t(1-k^2)}{\text{v}} = 0$$

$$k = \frac{1}{\sqrt{1-\text{v}^2/c^2}} \tag{10.21}$$

Substitution of this value of k in Eq. (10.12) gives the co-ordinate transformation equations:

$$x' = \frac{x - \text{v}t}{\sqrt{1-\text{v}^2/c^2}} \tag{10.22a}$$

$$y' = y \tag{10.22b}$$

$$z' = z \tag{10.22c}$$

$$t' = \frac{t - (\text{v}x/c^2)}{\sqrt{1-\text{v}^2/c^2}} \tag{10.22d}$$

Equation (10.22) is called the **Lorentz transformation** equations, which Lorentz derived in electromagnetism. Here, it is done on a more dynamical basis.

The inverse transformation can be obtained by interchanging the primed and unprimed quantities and reversing the sign of the relative velocity, since *S* and *S* differ only in the sign of the relative velocity. With the usual abbreviations

$$\beta = \frac{\text{v}}{c} \qquad \gamma = \frac{1}{\sqrt{1-\text{v}^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} \tag{10.23}$$

the Lorentz transformation and the inverse transformation simplify to

$$x' = \gamma(x - \beta ct) \qquad x = \gamma(x' + \beta ct') \tag{10.24a}$$

$$y' = y \qquad y = y' \tag{10.24b}$$

$$z' = z \qquad \text{or} \qquad z = z' \tag{10.24c}$$

$$t' = \gamma\left(t - \frac{\beta}{c}x\right) \qquad t = \gamma\left(t' + \frac{\beta}{c}x'\right) \tag{10.24d}$$

In the low velocity limit, where $b << 1$, it follows that the Lorentz transformation reduces to the Galilean transformation. It may be noted that the space transformation involves time and the time transformation involves the space coordinate. Hence, the transformation is sometimes referred to as **space-time transformation.**

Lorentz transformation sets a limit on the maximum value of v. If v > *c*, the quantity $\sqrt{1-\beta^2}$ becomes imaginary. The space and time coordinates would then become imaginary, which is physically unacceptable. Hence, in vacuum nothing can move with a velocity greater than the velocity of light.

10.6 VELOCITY TRANSFORMATION

Again, consider two inertial systems *S* and *S* moving with relative velocity v along the *xx*-axes. Consider a particle at *P* which is moving with a velocity *u* as measured by an observer in *S*. Its velocity as measured by an observer in *S* is *u*. The velocity components in *S* and *S* are

$$u_x = \frac{dx}{dt} \qquad u_y = \frac{dy}{dt} \qquad u_z = \frac{dz}{dt} \tag{10.25}$$

$$u'_x = \frac{dx'}{dt'} \qquad u'_y = \frac{dy'}{dt'} \qquad u'_z = \frac{dz'}{dt'} \tag{10.26}$$

Differentiation of the Lorentz transformation equation denoted by Eq. (10.22) gives:

$$dx' = \gamma(dx - \text{v}dt) \qquad dy' = dy$$

$$dz' = dz \qquad dt' = \gamma\left(dt - \frac{\text{v}dx}{c^2}\right) \tag{10.27}$$

Substitution of these values in Eq. (10.26) gives:

$$u'_x = \frac{dx - \text{v}dt}{dt - (\text{v}dx/c^2)} = \frac{(dx/dt) - \text{v}}{1 - (\text{v}/c^2)(dx/dt)}$$

$$u'_x = \frac{u_x - \text{v}}{1 - (\text{v}u_x/c^2)} \tag{10.28a}$$

$$u'_y = \frac{u_y}{\gamma(1 - \text{v}u_x/c^2)} = \frac{u_y\sqrt{1-\beta^2}}{1 - (\text{v}u_x/c^2)} \tag{10.28b}$$

$$u'_z = \frac{u_z}{\gamma(1 - \text{v}u_x/c^2)} = \frac{u_z\sqrt{1-\beta^2}}{1 - (\text{v}u_x/c^2)} \tag{10.28c}$$

These are the **Lorentz velocity transformations**. It may be noted that the velocity components u'_y and u'_z also depend on u_x. The inverse transformation is obtained by replacing v by –v and interchanging primed and unprimed coordinates:

$$u_y = \frac{u'_y}{\gamma[1 + (vu'_x / c^2)]} = \frac{u'_y \sqrt{1-\beta^2}}{1 + (vu'_x / c^2)} \quad (10.29b)$$

$$u_z = \frac{u'_z}{\gamma [1 + (vu'_x / c^2)]} = \frac{u'_z \sqrt{1-\beta^2}}{1 + (vu'_x / c^2)} \quad (10.29c)$$

If $\boldsymbol{u}'$ is along the x'-axis, $u'_x = u', u'_y = 0, u'_z = 0$. Then

$$u' = \frac{u - v}{1 - (uv / c^2)} \quad \text{or} \quad u = \frac{u' + v}{1 + (u'v / c^2)} \quad (10.30)$$

Equation (10.30) is referred to as *Einstein's law of addition of velocities*. Here, v is the velocity of frame *S* with respect to *S* and *u* is the velocity of the event *P* relative to *S* and *u* is the velocity of the event *P* relative to *S*. (See Fig. 10.4)

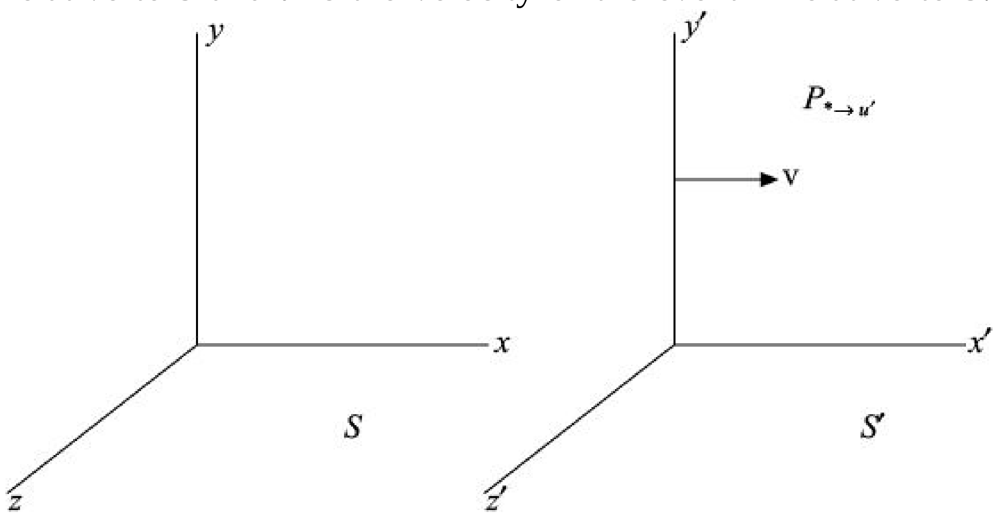


Fig. 10.4 Inertial frames of reference *S*, *S* and event at *P*.

If $u = c$, the velocity of light $u = \dfrac{c + v}{1 + \dfrac{v}{c}} = c$

That is, the velocity of the source does not add anything to the velocity of light emitted by it. In other words, it is impossible to exceed the velocity of light by adding two or more velocities, no matter how close each of these velocities to are that of light.

10.7 LENGTH CONTRACTION

Let S and S be inertial systems moving with relative velocity v along the xx axes. Consider a rod at rest in the inertial system S lying parallel to the x-axis. Though the system S is moving with a relative velocity, to the observer in S the rod is at rest. The length in an inertial frame in which the rod is at rest is called its **proper length**. The length of the rod $L_0 = x_2' - x_1'$, where x_1' and x_2' are the coordinates of its two ends measured at the same instant of time. To an observer in S, the length of the rod $L = x_2 - x_1$ where x_1 and x_2 are the coordinates of its two ends measured at the same time; therefore, $t_2 = t_1 = t$. Using Lorentz transformation

$$x_2' = \gamma(x_2 - vt_2) \qquad x_1' = \gamma(x_1 - vt_1)$$

Use of these expressions in $L_o = x_2' - x_1'$ gives

$$L_0 = \gamma(x_2 - x_1) = \gamma L$$

$$L = L_0\sqrt{1-\beta^2} \tag{10.31}$$

Since $\sqrt{1-\beta^2}$ is always less than unity, the length $L < L_0$. That is, to an observer in S the rod looks as though it is contracted parallel to the direction of motion.

The effect is reciprocal. If a rod has a length L_0. in S, to an observer in S which is in relative motion, it will appear to be of length $L_0\sqrt{1-\beta^2}$. The phenomenon of length contraction is referred to as **Lorentz-Fitzgerald contraction.** Thus, the space which is reduced to the measurement of length in physics and the geometrical shapes of objects cannot be absolute but only relative.

10.8 TIME DILATION

Consider two successive events occurring at the same point x in the inertial frame S. Let t_1' and t_2' be the times recorded by the observer in frame S. Then the time interval measured by him is $t_2' - t_1'$. For the observer in S, the rest frame is S itself. The time interval between events in the rest frame, that is, time interval as measured by a clock in S, is called the **proper time** Dt. Hence,

$$\Delta\tau = t_2' - t_1' \tag{10.32}$$

However, an observer in frame S measures these instants as

$$t_1 = \gamma[t_1' + (\mathrm{v}x'/c^2)] \qquad t_2 = \gamma[t_2' + (\mathrm{v}x'/c^2)]$$

The time interval according to the observer in frame S is then $\Delta t = t_2 - t_1$, which is given by

$$\Delta t = \gamma(t_2' - t_1') \qquad \text{or} \qquad \Delta t = \gamma\,\Delta\tau$$

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \mathrm{v}^2/c^2}} = \frac{\Delta\tau}{\sqrt{1-\beta^2}} \tag{10.33}$$

Since $g > 1$, it follows from Eq. (10.33) that the proper time interval is a minimum. The effect is known as **time dilation** and is equivalent to the slowing down of moving clocks. Hence, growth, aging, pulse rate, heartbeats, *etc.* are slowed down in a fast-moving frame. If the velocity of the moving frame $v = c, \Delta t \to \infty$ and the process of aging will stop altogether.

The time dilation effect has been verified experimentally by observation on elementary particles and by atomic clocks accurate to nanoseconds carried aboard jet planes.

10.9 SIMULTANEITY

Another important consequence of Lorentz transformation is that simultaneity is relative. Consider two events occurring at two different points x_1 and x_2 at times t_1 and t_2 in the inertial system S. Let t_1' and t_2' be the times at which the two events are observed to occur with respect to S. Then from Lorentz transformation

$$t_1' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right) \qquad t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$t_2' - t_1' = \gamma(t_2 - t_1) + \frac{\gamma v}{c^2}(x_1 - x_2) \tag{10.34a}$$

If the two events are occurring at the same instant in S, $t_2 - t_1 = 0$ and

$$t_2' - t_1' = \frac{\gamma v}{c^2}(x_1 - x_2) \neq 0 \tag{10.34b}$$

That is, two events that are simultaneous in one reference frame are not simultaneous in another frame of reference moving relative to the first, unless the two events occur at the same point in space. It implies that clocks that appear to be synchronized in one frame of reference will not necessarily be synchronized in another frame of reference in relative motion.

10.10 MASS IN RELATIVITY

In Newtonian mechanics, mass is considered to be a constant quantity independent of its velocity. In relativity, like length and time, it is likely to depend on its velocity u. That is, $m = m(u)$ and when $u = 0$, $m = m_0$, the **rest mass** of the particle. We now obtain the form of $m(u)$ by applying the law of conservation of linear momentum, which is a basic principle in physics, together with Lorentz velocity transformations.

Consider an inelastic collision between two identical bodies in the inertial system S which is moving relative to the inertial system S with a velocity v along the xx-axes. (See Fig. 10.5.) Assume that the identical bodies are moving in opposite directions along the x axis with velocities u and $-u$ in S. The masses of these bodies as observed from the system S be m_1 and m_2 and their velocities be $u_1$and u_2, respectively. In S the masses of the bodies are equal, and their momenta equal and oppositely directed. Hence, after the collision the two bodies will stick together and will be at rest in S. After collision, the mass $(m_1 + m_2)$ will be moving with velocity v in system S. Applying the law of conservation of linear momentum to the system S

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\text{v} \tag{10.35}$$

where according to Lorentz velocity transformations, u_1 and u_2 are

$$u_1 = \frac{u' + \text{v}}{1 + (u'\text{v}/c^2)} = \frac{u' + \text{v}}{1 + x} \tag{10.36a}$$

$$u_2 = \frac{-u' + \text{v}}{1 - (u'\text{v}/c^2)} = \frac{-u' + \text{v}}{1 - x} \tag{10.36b}$$

where,

$$x = u'\text{v}/c^2 \tag{10.37}$$

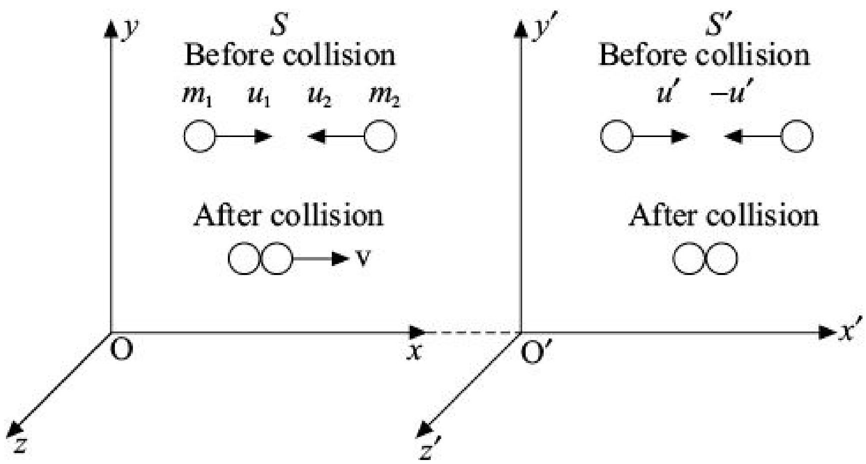


Fig. 10.5 Collision between two bodies taking place in the system *S* as observed from the system *S*.

Substituting the values of u_1 and u_2 in Eq. (10.35)

Substituting the values of u_1 and u_2 in Eq. (10.35)

$$m_1\left(\frac{u' + \text{v}}{1 + x}\right) + m_2\left(\frac{-u' + \text{v}}{1 - x}\right) = (m_1 + m_2)\,\mathbf{v}$$

$$\frac{m_1}{m_2} = \frac{1 + x}{1 - x} \tag{10.38}$$

Squaring Eq. (10.36a) and subtracting $\mathbf{c}^2$ from both sides

$$u_1^2 - c^2 = \frac{u'^2 + \text{v}^2 + 2c^2 x}{(1 + x)^2} - c^2 = \frac{u'^2 + \text{v}^2 - c^2 - c^2 x^2}{(1 + x)^2} \tag{10.39}$$

Similarly, from Eq. (10.36b) we have

$$u_2^2 - c^2 = \frac{u'^2 + v^2 - c^2 - c^2x^2}{(1-x)^2} \tag{10.40}$$

From Eqs. (10.39) and (10.40)

$$\frac{u_2^2 - c^2}{u_1^2 - c^2} = \frac{(1+x)^2}{(1-x)^2} \tag{10.41}$$

$$\frac{1+x}{1-x} = \frac{\sqrt{1 - u_2^2/c^2}}{\sqrt{1 - u_1^2/c^2}} \tag{10.42}$$

Combining Eqs.(10.38) and (10.42), we get

$$\frac{m_1}{m_2} = \frac{\sqrt{1 - u_2^2/c^2}}{\sqrt{1 - u_1^2/c^2}} \tag{10.43}$$

If the velocity of the second body as observed with respect to S is zero, then its mass m_2 is m_0, the rest mass of the identical body. This gives

$$m_1 = \frac{m_0}{\sqrt{1 - u_1^2/c^2}} \tag{10.44}$$

Thus, in general, if a mass is moving with a velocity **u** relative to an observer

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \tag{10.45}$$

In relativity, the invariant quantity is the rest mass m_0. In classical physics, m_0 is used in place of m also since the speeds acquired by objects are considerably small as compared to that of light. The momentum is defined by

$$\mathbf{p} = m\mathbf{u} = \frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}} \tag{10.46}$$

Newton's second law of motion takes the form

$$\mathbf{F} = \frac{d(m\mathbf{u})}{dt} = \frac{d}{dt}\left(\frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) \tag{10.47}$$

Thus, the change in the definition of mass has modified the definition of momentum and Newton's second law of motion.

10.11 MASS AND ENERGY

We see in this section how kinetic energy and total energy gets modified in relativity. Consider a particle of rest mass m_0 acted upon by a force **F** through a distance *x* in time *t* along the *x*-axis. Because of the force, the particle attains a final velocity **u**. Then the kinetic energy *T* is defined as

$$T = \int_0^x F dx = \int_o^x \frac{d}{dt}(mu) dx = \int_o^u \frac{dx}{dt} d(mu)$$

$$= \int_0^u u\, d(mu) = m_0 \int_0^u u\, d(\gamma u) \qquad (10.48)$$

where, $\gamma = \dfrac{1}{(1 - u^2/c^2)^{1/2}}$

Carrying out the differentiation

$$d(\gamma u) = \gamma du + u\, d\gamma = \frac{du}{(1 - u^2/c^2)^{1/2}} + \frac{u^2}{c^2} \frac{du}{(1 - u^2/c^2)^{3/2}}$$

$$= \frac{du}{(1 - u^2/c^2)^{3/2}} \qquad (10.49)$$

Substituting this value of $d(\gamma u)$ in Eq. (10.48) we have

$$T = m_0 \int_0^u \frac{udu}{(1 - u^2/c^2)^{3/2}} = m_0 c^2 \left[\frac{1}{(1 - u^2/c^2)^{1/2}} \right]_0^u$$

$$= \frac{m_0 c^2}{(1 - u^2/c^2)^{1/2}} - m_0 c^2 = mc^2 - m_0 c^2 \qquad (10.50)$$

Equation (10.50) looks very different from the classical expression ½ $m\mathbf{u}^2$. This relation implies that mass is a form of energy. Einstein called $m_0\mathbf{c}^2$, the **rest energy** of the object. It is the total energy of the object measured in a frame of reference in which the object is at rest. By analogy, mc^2, the sum of kinetic energy and rest energy, is called the **total energy** E: $E = mc^2$ (10.51) Equation (10.51) which states the relationship between mass and energy is Einstein's **mass-energy relation.**

The change of mass to other forms of energy and vice versa have been experimentally confirmed. This interconversion is easily detected in elementary particle physics. Electromagnetic radiation under certain conditions can be converted into electron and positron. The energy produced in nuclear power plants is a result of the loss in mass of the fuel during a fission reaction. Even the radiant energy we receive from the sun is an example of conversion of mass into energy.

A useful relation connecting the total energy E, momentum p and rest energy m_0c^2 can be obtained as detailed below. We have

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}}$$

Multiplying both sides by c^2 and squaring

$$m^2c^4\left(1 - \frac{u^2}{c^2}\right) = m_0^2c^4$$

$$m^2c^4 = m^2u^2c^2 + m_0^2c^4$$

Using the results $p = mu$ and $E = mc^2$, we get

$$E^2 = c^2p^2 + m_0^2\,c^4 \qquad (10.52)$$

which is the required relation. This allows us to examine the possibility of particles with zero rest mass. With $m_0 = 0$, Eq. (10.52) reduces to

$$E = cp \qquad (10.53)$$

When $\qquad m_0 = 0,\ mc^2 = muc \qquad \text{or} \qquad u = c$

Hence, massless particles must travel at the speed of light. Examples of particles in this category are photon, neutrino and graviton.

10.12 RELATIVISTIC LAGRANGIAN OF A PARTICLE

We shall try to find the relativistic Lagrangian and Hamiltonian of a single particle, which will give the spatial part of the the equations of motion in an inertial frame. The Lagrangian L is, in general, a function of the position coordinates x_i, the velocities $\dot{x}_i$ and time. In nonrelativistic mechanics, the generalized momentum components of a particle are defined by $p_i = (\partial L / \partial \dot{x}_i)$. For a single particle acted upon by conservative forces, assuming a similar definition for the momentum components in relativistic mechanics, one has

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{m_0 \dot{x}}{\sqrt{1 - u^2/c^2}} \qquad (10.54a)$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{m_0 \dot{y}}{\sqrt{1 - u^2/c^2}} \qquad (10.54b)$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = \frac{m_0 \dot{z}}{\sqrt{1 - u^2/c^2}} \qquad (10.54c)$$

where $u^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, u is the velocity of the particle in the inertial frame under consideration. The velocity-dependent part of L can be obtained by integrating Eq. (10.54). Integrating

$$L = -m_0 c^2 \sqrt{1 - u^2/c^2} - V \qquad (10.55)$$

where the constant of integration is taken as the potential function V which is a function of position coordinates.

Though the Lagrangian contains the potential function V as in nonrelativistic mechanics, the remaining in Eq. (10.55) is not equal to the kinetic energy. When

$u << c$ $$-m_0\, c^2 \left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}} \approx -m_0\, c^2 \left(1 - \frac{u^2}{2c^2}\right) = -m_0 c^2 + \frac{1}{2} m_0 u^2 \qquad (10.56)$$

The second term on the right resembles the kinetic energy term of nonrelativistic mechanics.

The form of the Lagrangian in Eq. (10.55) can be justified by obtaining Lagrange's equations of motion which are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad (10.57)$$

From Eq. (10.55), for the x-co-ordinate

$$\frac{\partial L}{\partial \dot{x}} = \frac{m_0 \dot{x}}{\sqrt{1-\beta^2}} \qquad \beta = \frac{u}{c} \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$$

Similar are the expressions for y—and z- coordinates. Substituting these values in Eq. (10.57), we have

$$\frac{d}{dt}\frac{m_0 \dot{x}}{\sqrt{1-\beta^2}} + \frac{\partial V}{\partial x} = 0 \qquad \beta = \frac{u}{C}$$

$$\frac{d}{dt} p_x = -\frac{\partial V}{\partial x} = F_x \qquad (10.58)$$

This agrees with the equation of motion of a particle, and therefore L given by Eq. (10.55) is the correct one in relativistic mechanics.

10.13 RELATIVISTIC HAMILTONIAN OF A PARTICLE

We can define the Hamiltonian of a system in a way similar to the definition in the nonrelativistic case $H = \sum_i \dot{q}_i p_i - L$

In the case of a single particle, this equation reduces to

$$H = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L$$

Using Eqs. (10.54) and (10.55)

$$H = \frac{m_0}{\sqrt{1-\beta^2}}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m_0c^2\sqrt{1-\beta^2} + V$$

$$= \frac{m_0u^2}{\sqrt{1-\beta^2}} + m_0c^2\sqrt{1-\beta^2} + V = \frac{m_0u^2 + m_0c^2(1-u^2/c^2)}{\sqrt{1-u^2/c^2}} + V$$

$$= \frac{m_0c^2}{\sqrt{1-u^2/c^2}} + V \qquad (10.59)$$

We have

$$E^2 = c^2p^2 + m_0^2c^4 \qquad \text{or} \qquad \frac{m_0^2c^4}{1-u^2/c^2} = c^2p^2 + m_0^2c^4$$

$$\frac{1}{1-u^2/c^2} = \frac{p^2}{m_0^2c^2} + 1$$

Substituting this value of $1/(1-u^2/c^2)$

$$H = m_0c^2\left(1 + \frac{p^2}{m_0^2c^2}\right)^{1/2} + V \qquad (10.60)$$

which is the Hamiltonian of a single particle. The two Hamilton's equations are

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i/m_0}{(1 + p^2/m_0^2c^2)^{1/2}} \qquad (10.61)$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i \qquad (10.62)$$

10.14 SPACE-TIME DIAGRAM

We have already adopted a notion of space-time as the setting in which physical events take place. The Lorentz transformation equation implies that space and time can no longer be considered independent entities. This four-dimensional world which is a linking together of space and time is called the **four-**

dimensional space-time continuum. Any four-dimensional space involving time in one of the axes is referred to as **four-space** or **world space**. The four-space with *ict* as the fourth coordinate is referred to as the **Minkowski four-space**. Thus, events are defined by 4 space-time coordinates and represented by points called **world points**. The position of a particle at different times represents a sequence of events and is called a **world line**.

For an event at (x, y, z, ict) in such a four-space, let a position vector or distance s be introduced such that the square of s has the form $s^2 = c^2 t^2 - x^2 - y^2 - z^2$ (10.63) We can easily prove that the quantity s^2 is invariant under the Lorentz transformation (See Worked Example 10.1). That is, $c^2 t^2 - x^2 - y^2 - z^2 = c^2 t^2 - x^2 - y^2 - z^2$ (10.64) Here (x, y, z, ict) are the four coordinates in the reference frame S and
(x, y, z, ict) are those in the frame S. Since the Lorentz transformation keeps the magnitude of the position vector in four-space constant, it is an orthogonal transformation. If (x_1, y_1, z_1, ict_1) and (x_2, y_2, z_2, ict_2) are the coordinates of any two events, then the quantity

$$s_{12} = \left[c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right]^{1/2} \quad (10.65)$$

is called the interval between these two events.

A full visualization of space-time requires a four-dimensional picture. We represent the situation in a two-dimensional plane by suppressing the two coordinates x_2 and x_3. Consider a particle moving with a uniform velocity u along the x_1-axis. This can be represented by the straight line OA (See Fig.10.6) having a slope $\tan^{-1}(u/c)$ with the ct-axis (the x_4-axis). This is understandable since $x = ut = (u/c)\, ct$. The line OA is the world line of the particle. For a light ray $u = c$ and therefore $\tan^{-1} u/c = \tan^{-1} 1 = 45°$. Hence, the path of a light ray is represented by the straight lines OB and OC inclined at angles of 45° to the coordinate axis in Fig. 10.6 (b). These lines are defined by the equation

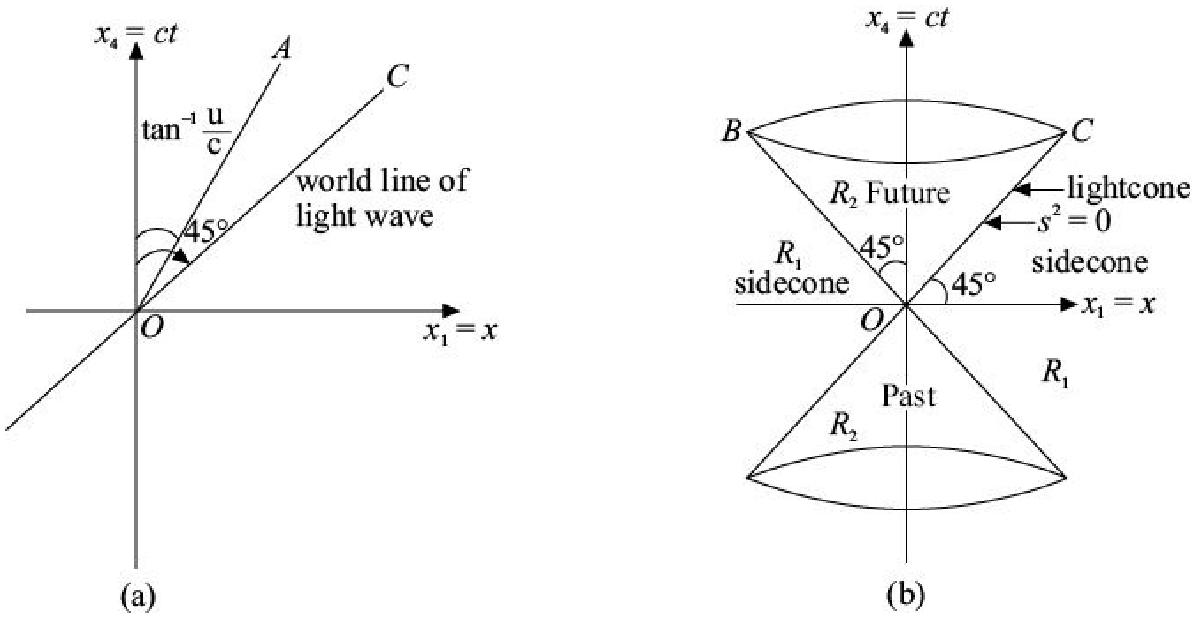


Fig. 10.6 (a) World line of light wave (b) light cones.

$s^2 = c^2t^2 - x^2 = 0$ (10.66) If we restore x_2 and x_3 this equation becomes $s^2 = c^2t^2 - x^2 - y^2 - z^2 = 0$ (10.67) where s is the interval between the origin and the point (x, y, z, ict). The interval for all points on the world line of a light ray vanishes, and the line in two dimensions is called a **null line** and in three dimensions it becomes a cone with apex at O, called a **null cone**. The value of $s^2 = 0$ on the surface of this cone. World lines of material particles lying within this cone must pass through the origin. The null cone constitutes the space-time representation of light and hence it is also called a **light cone**.

The light cone divides the four-dimensional space into two regions characterized by the inequalities: Region $R_1 : s^2 = c^2\, t^2 - x^2 - y^2 - z^2 < 0$ (10.68) Region $R_2 : s^2 = c^2\, t^2 - x^2 - y^2 - z^2 > 0$ (10.69) As already mentioned, $s^2 = 0$ on the surface of the light cone. The intervals are said to be **space-like**, **time-like** or **light (null)-like** according to whether s^2 is less than, greater than or equal to zero. Events lying inside the upper and lower cones are separated from the origin by time-like intervals ($s^2 > 0$). In all frames of reference, in the upper cone we have events which lie after the event at the origin. Hence, this region is referred to as **absolute future**. In the lower cone, we have the events which preceded the event at the origin and hence it is referred to as the **absolute past**. All events in the two side cones are separated from the event at the origin by space-like intervals ($s^2 < 0$).

10.15 GEOMETRICAL INTERPRETATION OF LORENTZ TRANSFORMATION

A geometrical interpretation of Lorentz transformation was given by Minkowski in analogy with the transformation of cartesian coordinates under spatial rotation. For anticlockwise rotation of cartesian coordinates in two dimensions through an angle *f* about the z-axis, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (10.70)$$

where (*x*, *y*) and (*x*, *y*) are the coordinates of a point before and after rotation. In the space-time case, consider a rotation of the x_1 x_4 plane through an angle *q* as shown in Fig. 10.7 where (x_1, x_2, x_3, x_4) stands for (*x*, *y*, *z*, *ict*). The coordinates of the point *P* before and after rotation are related by

$$\begin{pmatrix} x_1' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x_1' &= x_1 \cos\theta + x_4 \sin\theta \\ x_4' &= -x_1 \sin\theta + x_4 \cos\theta \end{aligned} \qquad (10.71)$$

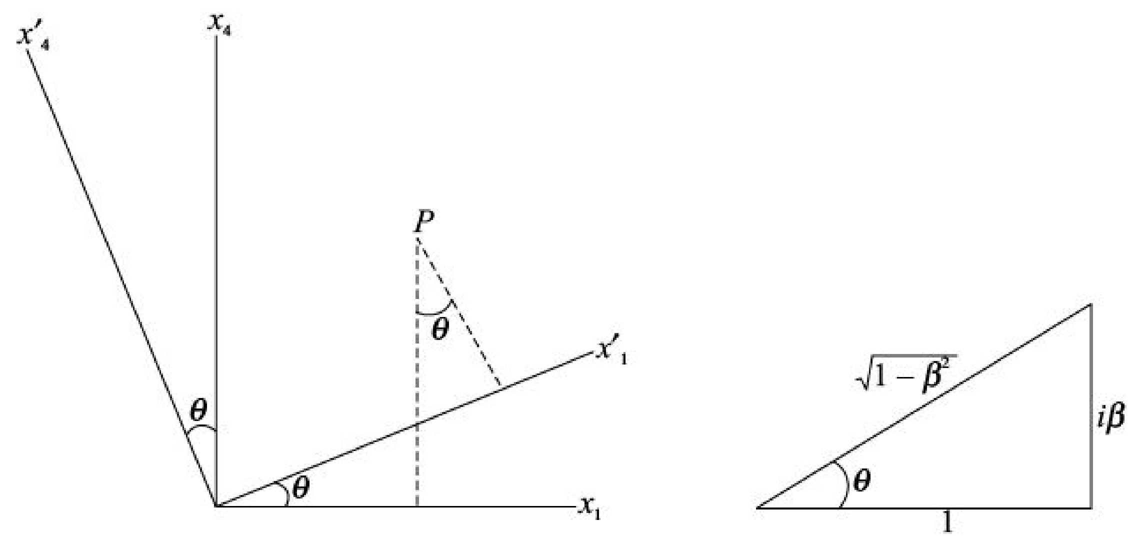


Fig. 10.7 Rotation of x_1 x_4 plane through an angle *q*.

If we set $\tan\theta = i\mathbf{v}/\mathbf{c} = i\beta$, where $\beta = \mathbf{v}/\mathbf{c}$

$$\cos\theta = \frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad \sin\theta = \frac{i\beta}{\sqrt{1-\beta^2}} \tag{10.72}$$

Substituting the values of $\cos\theta$ and $\sin\theta$ in Eq.(10.71) and setting

$x_1 = x, x_1' = x',\ x_4 = ict,\ x_4' = ict'$, we get

$$x' = \frac{x}{\sqrt{1-\beta^2}} - \frac{\mathbf{v}t}{\sqrt{1-\beta^2}} = \frac{x - \mathbf{v}t}{\sqrt{1-\mathbf{v}^2/c^2}} \tag{10.73}$$

$$t' = \frac{t - (\mathbf{v}x/c^2)}{\sqrt{1-\beta^2}} = \frac{t - (\mathbf{v}x/c^2)}{\sqrt{1-\mathbf{v}^2/c^2}} \tag{10.74}$$

Equations (10.73) and (10.74) together with $y = y$ and $z = z$ are the same as the Lorentz transformation equation denoted by Eq. (10.22). Thus, the Lorentz transformation can be regarded as a rotation in the $x_1\ x_4$ plane of Minkowski space through an imaginary angle defined by Eq. (10.72).

10.16 PRINCIPLE OF COVARIANCE

The postulate of equivalence requires that the mathematical equations representing the physical laws should be covariant, *co* means the same. If the equation in one inertial frame is $A = B$, then that in another frame should be $A = B$. Here A and B may both be scalars, vectors, tensors or any other geometrical object. In other words, if a particular component of A is multiplied by M while going from one coordinate system to another, the corresponding component of B should also be multiplied by the same factor. The equation is said to be covariant as both sides vary in the same manner. This principle is called the **principle of covariance**.

We have already seen that the length of a rod contracts and time does not remain invariant. The quantity that remains invariant is $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$

Therefore, one must use four-dimensional vectors whose norms $a_1^2 + a_2^2 + a_3^2 + a_4^2$, a_4 being imaginary, remain invariant. Such vectors transform

according to the rule $a'_\alpha = C_{\alpha\beta}\, a_\beta$ (10.75) Here we use *Einstein's convention*, according to which a repeated index indicates that we have to take the sum over all the possible values of the index.
In this case the index *b* is repeated, so we have to sum over all values 1 to 4, the possible values for *a* and *b*. However, when we use Latin indices the summation is for 1 to 3.

10.17 FOUR-VECTORS IN MECHANICS

A vector in four-dimensional Minkowski space is called a four-vector, if its components transform under a Lorentz transformation in the same way as the x_1, x_2, x_3, x_4 coordinates of a point. For example, $A_m = (A_1, A_2, A_3, A_4)$ is defined to be a four-vector if, under a Lorentz transformation,

$$A_1' = \gamma(A_1 + i\beta A_4); \qquad A_2' = A_2, \qquad A_3' = A_3, \qquad A_4' = \gamma(A_4 - i\beta A_1) \tag{10.76}$$

Equation (10.76) can be put in the matrix form as

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{10.77}$$

Here prime refers to the S' frame of reference. Now

$$\begin{aligned} A_1'^2 + A_2'^2 + A_3'^2 + A_4'^2 &= \gamma^2\,(A_1 + i\beta A_4)^2 + A_2^2 + A_3^2 + \gamma^2\,(A_4 - i\beta A_1)^2 \\ &= \gamma^2(1-\beta^2)\,A_1^2 + A_2^2 + A_3^2 + \gamma^2(1-\beta^2)A_4^2 \\ &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \end{aligned} \tag{10.78}$$

That is, the length of a four-vector is unchanged under a Lorentz transformation which is equivalent to a rotation of axes.

Position Four-Vector

A position four-vector, written as **X**, can be represented by the components X_1, X_2, X_3, X_4 as

$$X_\mu = (X_1, X_2, X_3, X_4) = (\mathbf{x}, x_4) = (x, y, z, ict) \tag{10.79}$$

where $\mathbf{x}$ stands for the usual three-dimensional position vector with components x, y, z. From Eq. (10.76) it is evident that the differential of $\mathbf{X}$ is also a four-vector given by

$$dX_\mu = (d\mathbf{x}, dx_4) = (dx, dy, dz, icdt) \tag{10.80}$$

and, $$dX_1'^2 + dX_2'^2 + dX_3'^2 + dX_4'^2 = dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2$$

This corresponds to the relation

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 = \text{invariant} \tag{10.81}$$

Four-Velocity

Let two events, having coordinates X_1, X_2, X_3, X_4 and $X_1 + dX_1, X_2 + dX_2, X_3 + dX_3, X_4 + dX_4$, respectively, refer to the positions of a particle at times t and $t + dt$ in the inertial frame S. If $\mathbf{u}$ is the three-dimensional velocity of the particle in the inertial frame S,

$$\frac{dx}{dt} = u_x \qquad \frac{dy}{dt} = u_y \qquad \frac{dz}{dt} = u_z \tag{10.82}$$

In the four-dimensional world, time t is a coordinate and is not an invariant quantity. An invariant parameter that can be considered for use is the **proper time interval** *dt* which is defined by the relation

$$d\tau^2 = -\frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \tag{10.83}$$

$$d\tau = dt\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]}$$

$$= dt\sqrt{1 - u^2/c^2}, \qquad u^2 = u_x^2 + u_y^2 + u_z^2 \tag{10.84}$$

Proper time interval *dt* is the time recorded by a clock moving with the particle (see Section 10.8). Since ds^2 is an invariant quantity, *dt* is also invariant. Hence, for the four-velocity *V*, we have

$$V_\mu = \frac{dx_\mu}{d\tau}, \qquad \mu = 1, 2, 3, 4 \tag{10.85}$$

$$= \left(\frac{dX_1}{d\tau}, \frac{dX_2}{d\tau}, \frac{dX_3}{d\tau}, \frac{dX_4}{d\tau} \right) = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, ic\frac{dt}{d\tau} \right)$$

Using Eqs. (10.82) and (10.84)

$$\frac{dx}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau} = \frac{u_x}{\sqrt{1 - u^2/c^2}}$$

$$\frac{dy}{d\tau} = \frac{u_y}{\sqrt{1 - u^2/c^2}}, \qquad \frac{dz}{d\tau} = \frac{u_z}{\sqrt{1 - u^2/c^2}}$$

Consequently,

$$V_\mu = \left(\frac{u_x}{\sqrt{1 - u^2/c^2}}, \frac{u_y}{\sqrt{1 - u^2/c^2}}, \frac{u_z}{\sqrt{1 - u^2/c^2}}, \frac{ic}{\sqrt{1 - u^2/c^2}} \right)$$

$$= \left(\frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}, \frac{ic}{\sqrt{1 - u^2/c^2}} \right) \tag{10.86}$$

Momentum Four-Vector

The rest mass of a particle m_0 has the same value when it is at rest in all inertial frames. Multiplying the four-velocity by the invariant rest mass m_0, one gets

$$P_\mu = (P_1, P_2, P_3, P_4) = m_0 V_\mu$$

$$P_\mu = \left(\frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}}, \frac{im_0 c}{\sqrt{1 - u^2/c^2}} \right) \tag{10.87}$$

Here, P_m is the four-vector momentum or the **four-momentum**. Since

$$m = m_0/\left(1 - u^2/c^2\right)^{\frac{1}{2}}$$

$$P_1 = m_0 u_1 = \frac{m_0 u_x}{\sqrt{1-u^2/c^2}} = m u_x = p_x$$

$$P_2 = m_0 u_2 = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} = m u_y = p_y$$

$$P_3 = m_0 u_3 = \frac{m_0 u_z}{\sqrt{1-u^2/c^2}} = m u_z = p_z$$

$$P_4 = \frac{i m_0 c}{\sqrt{1-u^2/c^2}} = imc = \frac{imc^2}{c} = \frac{iE}{c}$$

Hence, writing **p** for p_x, p_y, p_z

$$P_\mu = \left(p_x,\ p_y,\ p_z,\ imc\right) = \left(\mathbf{p}, \frac{\mathrm{iE}}{c}\right) \qquad (10.88)$$

Obviously, the fourth component of the momentum four-vector is proportional to the total energy of the particle. As the length of a four-vector is invariant

$$p^2 + \left(\frac{iE}{c}\right)^2 = \text{constant} \qquad \text{or} \qquad p^2 - \frac{E^2}{c^2} = \text{constant} \qquad (10.89)$$

When $p = 0$, $E = m_0 c^2$. Hence, the constant in Eq. (10.89) is $-m_0^2 c^2$. Consequently $E^2 = c^2 p^2 + m_0^2 c^4$, which is Eq. (10.52).

Four-Force

From Eq. (10.88), it follows that

$$dP_\mu = \left(d\mathbf{p}, \frac{idE}{c} \right) \tag{10.90}$$

is a four-vector. Dividing both sides of Eq. (10.90) by $d\tau$, one gets the four-vector force or **four-force** which is given by

$$F_\mu = \frac{dP_\mu}{d\tau} = \left(\frac{d\mathbf{p}}{d\tau}, \frac{i}{c}\frac{dE}{d\tau} \right) \tag{10.91}$$

$$\frac{d\mathbf{p}}{d\tau} = \frac{d\mathbf{p}}{dt} \cdot \frac{dt}{d\tau} = \mathbf{f}\, \frac{1}{\sqrt{1-u^2/c^2}} \tag{10.92}$$

$$\frac{dE}{d\tau} = \frac{dE}{dt} \cdot \frac{dt}{d\tau} = \frac{d}{dt}(T + m_0c^2) \cdot \frac{1}{\sqrt{1-u^2/c^2}}$$

$$= \frac{dT}{dt} \frac{1}{\sqrt{1-u^2/c^2}} \tag{10.93}$$

where Eq. (10.84) is used. Work done by a force $\mathbf{f}$ in a displacement $d\mathbf{l} = \mathbf{f} \cdot d\mathbf{l}$ Equating the work done to the increase in kinetic energy

$$dT = \mathbf{f} \cdot d\mathbf{l} \qquad \text{or} \qquad \frac{dT}{dt} = \mathbf{f} \cdot \frac{d\mathbf{l}}{dt} = \mathbf{f} \cdot \mathbf{u}$$

Hence,

$$\frac{i}{c}\frac{dE}{d\tau} = \frac{i}{c}\frac{1}{\sqrt{1-u^2/c^2}}\frac{dT}{dt} = \frac{i}{c}\frac{\mathbf{f}.\mathbf{u}}{\sqrt{1-u^2/c^2}} \tag{10.94}$$

Consequently,

$$F_\mu = \left(\frac{\mathbf{f}}{\sqrt{1-u^2/c^2}}, \frac{i}{c}\frac{\mathbf{f}\cdot\mathbf{u}}{\sqrt{1-u^2/c^2}} \right) \tag{10.95}$$

Four-Acceleration

One can get the **four-acceleration** A_μ easily from Eq. (10.86)

$$A_\mu = \frac{dV_\mu}{d\tau} = \left(\frac{d}{d\tau}\left(\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}} \right), \qquad ic\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-u^2/c^2}} \right) \right) \tag{10.95a}$$

10.18 CHARGE CURRENT FOUR-VECTOR

It is an established experimental fact that the total charge in a system is not altered by the motion of its carrier. Hence, one can state that the total charge in an isolated system is relativistically invariant. Consider a volume element $dV = dx_1 \; dx_2 \; dx_3$ with charge dq in it. Then the charge density

$$\rho = \frac{dq}{dV} \qquad \text{or} \qquad dq = \rho \, dx_1 \, dx_2 \, dx_3$$

Multiplying both sides by dx_μ

$$\begin{aligned} dq \, dx_\mu &= \rho \frac{dx_\mu}{dt} dx_1 \, dx_2 \, dx_3 \, dt \\ &= \rho \frac{dx_\mu}{dt} dx_1 \, dx_2 \, dx_3 \frac{d(ict)}{ic} \\ &= \rho \frac{dx_\mu}{dt} dx_1 \, dx_2 \, dx_3 \frac{dx_4}{ic} \end{aligned} \tag{10.96}$$

The quantity $dx_1 \, dx_2 \, dx_3 \, dx_4/ic$ is Lorentz-invariant and hence it is a scalar. Since the charge is invariant, the left hand side of Eq. (10.96), $dq \, dx_\mu$ is a four-vector. Consequently,

$$J_\mu = \rho \frac{dx_\mu}{dt}$$

will be a four-vector called the **current four-vector**. Now

$$J_1 = \rho \frac{dx_1}{dt} = \rho u_1; \qquad J_2 = \rho u_2; \qquad J_3 = \rho u_3 \tag{10.97a}$$

$$J_4 = \rho \frac{dx_4}{dt} = \rho \frac{d(ict)}{dt} = ic\rho \tag{10.97b}$$

Hence, the four-vector

$$J_\mu = (J_1, J_2, J_3, J_4) = (j_x, j_y, j_z, ic\rho)$$

$$J_\mu = (\mathbf{j}, ic\rho) \tag{10.98}$$

where $\mathbf{j}$ is the current density with components j_x, j_y, j_z. The charge-current equation of continuity in electrodynamics

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \tag{10.99}$$

can now be expressed in the covariant form as

$$\frac{\partial J_\mu}{\partial x_\mu} = 0 \tag{10.100}$$

This relation establishes the Lorentz covariance of the charge-current conservation relation.

10.19 INVARIANCE OF MAXWELL'S EQUATIONS

Maxwell's Equations

The classical theory of radiation is based on Maxwell's equations for the electromagnetic field. The two basic quantities describing the electromagnetic field are the electric and magnetic field strengths **E** and **B** which are functions of space and time. Maxwell's equations in free space are

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho \tag{10.101}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{10.102}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{10.103}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \tag{10.104}$$

where ε_0 and μ_0 are respectively the permittivity and permeability of free space. These coupled first order differential equations can be solved to get **E** and **B**, which gives a complete description of the electromagnetic field.

Vector and Scalar Potentials

Instead of **E** and **B**, the field equations can also be conveniently expressed in terms of a vector potential **A** and a scalar potential *f*. Eq. (10.102) implies that **B** can be written as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{10.105}$$

With this definition of **B**, Eq. (10.103) takes the form

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{10.106}$$

Since the curl of the gradient of a scalar function is zero, from Eq. (10.106) we have

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi \tag{10.107}$$

where ϕ is the scalar potential. From Eq. (10.107)

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \tag{10.108}$$

which gives the electric field in terms of the potentials **A** and *f*.

The other two equations, Eqs. (10.101) and (10.104), can also be expressed in terms of **A** and *f*. Substituting the value of **E** in Eq. (10.101)

$$\nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) = \frac{\rho}{\varepsilon_0}$$

$$\nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} \tag{10.109}$$

Substitution of Eqs. (10.105) and (10.108) in Eq. (10.104) and the use of the result $\varepsilon_0 \mu_0 = 1/\mathbf{c}^2$ gives

$$\nabla \times (\nabla \times \mathbf{A}) - \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) = \mu_0 \mathbf{j}$$

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathrm{A}}{\partial t^2} + \frac{1}{c^2} \nabla \cdot \frac{\partial \phi}{\partial t} = \mu_0 \mathbf{j}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{j} \tag{10.110}$$

The solution of Maxwell's equations is thus reduced to solving the coupled equations, Eqs. (10.109) and (10.110), for **A** and *f*.

Gauge Transformations

The potentials **A** and *f* as defined above are not unique. We now use a property of classical electrodynamics, called **gauge invariance**, to decouple the two equations. The transformations

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi \qquad (10.111)$$

$$\phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t} \qquad (10.112)$$

where χ is an arbitrary scalar function, will leave **B** and **E** unchanged. The fact that $\nabla \times \nabla\chi = 0$ leaves **B** unchanged by the change $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$. The changes in **A** and ϕ do not affect the electric field, since Eq. (10.108) changes to

$$\begin{aligned} \mathbf{E} &= -\nabla\left(\phi - \frac{\partial \chi}{\partial t}\right) - \frac{\partial(\mathbf{A} + \nabla\chi)}{\partial t} \\ &= -\nabla\phi + \nabla\frac{\partial \chi}{\partial t} - \frac{\partial \mathrm{A}}{\partial t} - \nabla\frac{\partial \chi}{\partial t} \\ &= -\nabla\phi - \frac{\partial \mathrm{A}}{\partial t} \end{aligned} \qquad (10.113)$$

That is, the gauge transformations to $(\mathbf{A}', \phi')$ leads to the same **E** and **B** as before and therefore the transformations leave Maxwell's equations invariant. One can use this invariance to select the family of potentials $(\mathbf{A}, \phi)$ such that

$$\nabla . A + \frac{1}{\mathbf{c}^2}\frac{\partial \phi}{\partial t} = 0 \qquad (10.114)$$

The condition in Eq. (10.114) removes the coupling of the two equations, Eqs. (10.109) and (10.110).

The freedom available in the definitions in Eqs. (10.111) and (10.112) together is called **gauge transformations,** and the condition in Eq. (10.114) is known as the **Lorentz gauge condition**. In view of this condition, Eqs. (10.109) and (10.110) reduce to

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_o} \tag{10.115}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} \tag{10.116}$$

Four-Vector Potential

The differential operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ on the left hand side of Eqs. (10.115) and (10.116) is referred to as the **d'Alembertian operator** and is denoted by the symbol $\Box^2$. This operator is Lorentz invariant (See Example 10.10). In terms of the operator $\Box^2$, Eqs. (10.116) and (10.115) reduce to

$$\Box^2\mathbf{A} = -\mu_0\mathbf{j} \tag{10.117}$$

$$\Box^2\left(\frac{i}{c}\phi\right) = -\frac{i\rho\mu_0}{c\varepsilon_0\mu_0} = -\mu_0 ic\rho \tag{10.118}$$

where we have used the relation $c^2 = 1/e_0m_0$. We have already defined the charge current four-vector as

$$J_\mu = (\mathbf{j},\ ic\rho)$$

We may therefore write the two equations, Eqs. (10.117) and (10.118), together in the form of the four-vector relation

$$\Box^2 A_\mu = -\mu_0 J_\mu \tag{10.119}$$

Since J_μ is a four-vector and $\Box^2$ is a four-dimensional scalar, A_μ must be a four-vector, called **four-vector potential**, defined as

$$A_\mu = (A_1, A_2, A_3, A_4) = \left(\mathbf{A}, \frac{i\phi}{c}\right) \tag{10.120}$$

Equation (10.119) gives Maxwell's equations in the four-vector form, which implies the invariance of Maxwell's equations. The Lorentz gauge condition, Eq. (10.114), can now be expressed in terms of A_m. From Eq. (10.114)

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial (i\phi/c)}{\partial (ict)} = 0$$

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 \tag{10.121}$$

which reduces to the following in the covariant form:

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{10.122}$$

The transformation equations for the electromagnetic potentials **A** and ϕ from the inertial frame S to inertial frame S' are

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ i\phi'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 1 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ i\phi/c \end{pmatrix} \tag{10.123}$$

10.20 ELECTROMAGNETIC FIELD TENSOR

The electric and magnetic field strengths of the electromagnetic field are expressed in terms of the vector potential **A** and scalar potential *f*. From Eq. (10.105)

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \tag{10.124a}$$

$$B_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \tag{10.124b}$$

$$B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \tag{10.124c}$$

From Eq. (10.108)

$$E_1 = -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial x} \qquad \text{or} \qquad \frac{i}{c}E_1 = \frac{\partial A_1}{\partial (ict)} - \frac{\partial (i\phi/c)}{\partial x_1}$$

$$\frac{i}{c}E_1 = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} = F_{41} \text{ (say)} \tag{10.125a}$$

$$\frac{i}{c}E_2 = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2} = F_{42} \tag{10.125b}$$

$$\frac{i}{c}E_3 = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3} = F_{43} \tag{10.125c}$$

In general, the components of **F** are given by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \qquad F_{\mu\nu} = -F_{\nu\mu} \qquad F_{\mu\mu} = 0 \tag{10.126}$$

These are the components of an antisymmetric tensor $F_{\mu\nu}$ defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{pmatrix} \tag{10.127}$$

which is the **electromagnetic field four-tensor**.

10.21 GENERAL THEORY OF

RELATIVITY

The special theory of relativity requires the modification of the classical laws of motion. The laws of electromagnetism, including the Lorentz force law, remain valid in relativity also. Though Newton's law of gravitation is successful in explaining a number of phenomena, conceptually it is found to be inadequate. The gravitational force of attraction between bodies is assumed to be transmitted instantaneously, that is, with infinite speed. This is in contradiction to the relativistic requirement that the limiting speed of a signal is the velocity of light. We have learnt that the laws of physics are the same in all inertial frames and that only the relative motion of a system with respect to another can be considered a physical reality. The generalization of the special theory of relativity to noninertial reference frames by Einstein in 1916 is known as the *general theory of relativity*.

Principle of Equivalence

The basis for the general theory is the principle of equivalence, which states that *a homogeneous gravitational field is completely equivalent to a uniformly accelerated reference frame.* Consider two reference frames: (i) a non-accelerating (inertial) reference frame S in which there is a uniform gravitational field; and

(ii) a reference frame S which is accelerating uniformly with respect to an inertial frame but in which there is no gravitational field. Two such frames are physically equivalent. That is, experiments carried out under otherwise identical conditions in these two frames should give the same results. This is Einstein's **principle of equivalence**.

The principle of equivalence is related to the concept of two types of mass: gravitational mass and inertial mass. Newton's law of gravitation states that one body attracts another body due to the gravitational force, and the strength of the force is proportional to the product of the masses of the two bodies: $F = \frac{GMm_G}{r_e^2}$

(10.128) where m_G is the gravitational mass of the object, M is the gravitational mass of the earth, r_e is the radius of the earth and G the gravitational constant. The **gravitational mass** measures how strongly an object is attracted to other masses. The other type of mass is the inertial mass. Newton's second law states that $F = m_I a$ (10.129) where m_I is the mass of the object or more precisely, the inertial mass. **Inertial mass** m_I measures how strongly an object resists a change in its motion. No experiment has been able to indicate any measurable difference between inertial and gravitational mass. Experiments have proved this result to better than one part in 10^{12}. Hence, this may be taken as another way to state the principle of equivalence: *gravitational mass is equivalent to inertial mass*.

Bending of Light in a Gravitational Field

An important prediction of the general theory of relativity is that light is affected by gravity. One of the basic properties of light is that it propagates along a straight line. However, a prediction of Einstein's theory is that the positions of stars whose light passes near the edge of the sun should be displaced due to deflection by the gravitational field of the sun. Fig. 10.8 (a) illustrates the deflection of light from a star in the gravitational field of the sun. The speed of light with mass $E/\mathbf{c}^2$ is reduced in the vicinity of the mass M, thus bending the beam. A calculation of this deflection gives 1.75 seconds for the net deflection of star light grazing the edge of the sun. A measurement could be made only during a total solar eclipse; otherwise the light from the stars would be lost in the brilliant sunshine. An opportune eclipse occurred in 1919 and the experimental results were compatible with Einstein's predictions.

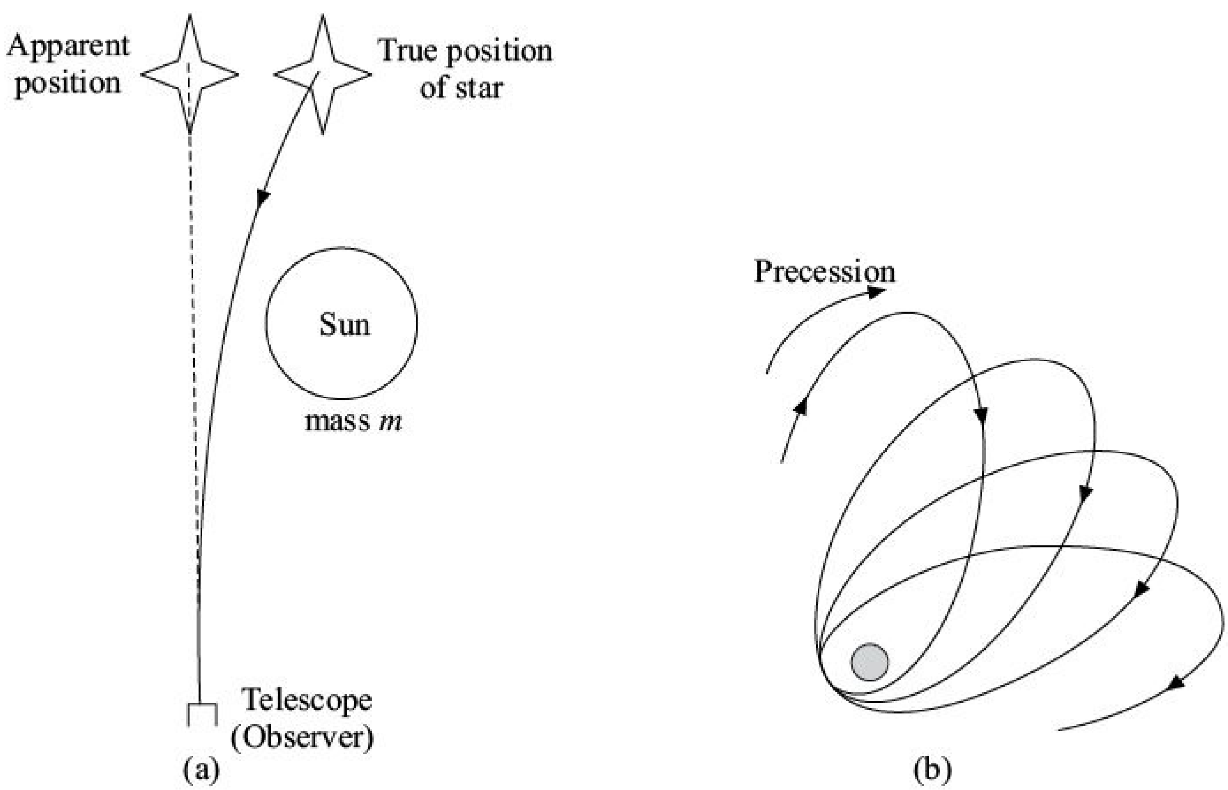


Fig. 10.8 (a) Deflection (not to scale) of a beam of star light due to the gravitational attraction of the sun; (b) Precession of an elliptical orbit.

Precession of the Perihelion of Planetary Orbits

As per Einstein's theory, the orbits obtained for planets are very similar to the ellipses of classical theory. However, the ellipse precesses very slowly in the plane of the orbit, so that the perihelion is at a slightly different angular position for each orbit, as shown in Fig. 10.8 (b). This shift is greatest in the case of the planet Mercury, which is close to the sun and hence in a very strong gravitational field. The perihelion advance of Mercury is predicted to be 43 second of arc in a century. This agrees with the discrepancy between classical theory and observation, which was known for many years before the advent of general theory of relativity.

Space Curvature

A light beam must travel by the shortest path between two points. We have already seen that it travels by a curved path. That is, if a light beam follows a curved path, then that curved path must be the shortest distance between the two points. This suggests that space itself is curved and the gravitational field is the one that causes the curvature. Indeed, the curvature of space or of four-dimensional space-time is a basic aspect of general relativity.

Figures drawn on plane surfaces are governed by the rules of classical Euclidean geometry whereas those on curved surfaces are not. For example, in plane geometry the sum of the angles of a triangle is 180°. To construct a triangle on a curved surface, say a sphere, consider the large triangle (See Fig. 10.9a) with one vertex at the pole and two others *B* and *C* on the equator. Since the meridians forming two sides of that triangle make 90° with the equator, the sum of the angles *ABC* and *ACB* is equal to 180°. In addition, we have the angle *BAC*. Hence, the sum is always greater than 180°, if the curved surface is a sphere. However, if a triangle is drawn on a saddle-like surface, the sum of the angles of the triangle will be less than 180°, as shown in Fig 10.9 (b). In this case the surface sinks between the vertices of the triangle, whereas in the former the surface bulges up between the vertices. In these cases the sides of the triangles are not straight lines in the usual sense. They represent the shortest distances between two given points and are called **geodesics**. In the geometry of curved surfaces, geodesics play the same role as that of straight lines in plane geometry. The curved surface of the sphere is said to have *positive curvature* since the surface always lies on one side of the tangent plane to the surface at a point. The saddle-like surface is said to have *negative curvature* as the surface and the tangent plane at a point intersect.

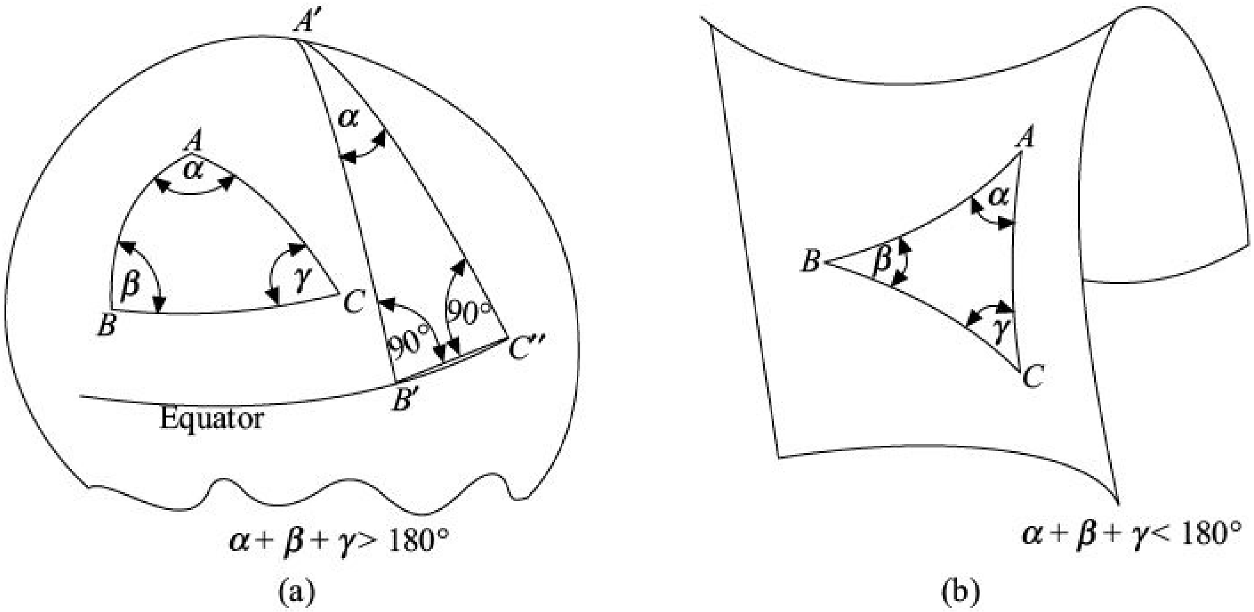


Fig. 10.9 Sum of angles of a triangle on a two-dimensional curved surface: (a) Positive curvature surface; (b) Negative curvature surface.

Gravitational Red Shift

Electromagnetic radiation of a given frequency emitted in a gravitational field will appear to an outside observer to have a lower frequency, that is, it will be **red shifted**. Consider photons emitted from the surface of a star of mass M_S, radius R_S and observed on earth. As the gravitational field of the star acts on the photon in an opposite direction, the photon loses energy, resulting in a decrease in frequency. The effective mass of the photon is $h\nu/c^2$ and therefore the decrease in energy is

$$\cong \frac{GM_s}{R_s}\left(\frac{h\nu}{c^2}\right)$$

Hence, the observed frequency ν' is given by

$$h\nu' \approx h\nu - \frac{GM_s}{R_s}\frac{h\nu}{c^2}$$

$$\nu' \approx \nu\left(1 - \frac{GM_s}{R_s c^2}\right) \tag{10.130}$$

A frequency decrease means wavelength increase, and we say that visible light is *shifted to red*.

WORKED EXAMPLES

Example 10.1 Show that $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformation.

Solution: Replacing x, y, z, t by x', y', z', t'

$$x^2 + y^2 + z^2 - c^2t^2 = \gamma^2 (x' + \text{v}t')^2 + y'^2 + z'^2 - c^2\gamma^2\left[t' + \left(\text{v}x'/c^2\right)\right]^2$$

$$= x'^2\gamma^2\left(1 - \frac{\text{v}^2}{c^2}\right) + y'^2 + z'^2 - t'^2\gamma^2c^2\left(1 - \frac{\text{v}^2}{c^2}\right)$$

$$= x'^2 + y'^2 + z'^2 - c^2t'^2$$

That is, $x^2 + y^2 + z^2 - c^2\, t^2$ is invariant under Lorentz transformation.

Example 10.2 A rocket leaves the earth at a speed of 0.6*c*. A second rocket leaves the first at a speed of 0.9*c* with respect to the first. Calculate the speed of the second rocket with respect to earth if: (i) it is fired in the same direction as the first one; (ii) it is fired in a direction opposite to the first.

Solution: (i) $u = \dfrac{u' + \text{v}}{1 + \dfrac{u'\text{v}}{c^2}} = \dfrac{0.9c + 0.6c}{1 + \dfrac{(0.9c)\ (0.6c)}{c^2}} = 0.974c$

(ii) $u = \dfrac{-0.9c + 0.6c}{1 - \dfrac{(0.9c)\ (0.6c)}{c^2}} = -\dfrac{0.3c}{0.46} = -0.652c$

Example 10.3 The length of a spaceship is measured to be exactly half its proper length. What is (i) the speed of the spaceship relative to the observer on earth? (ii) the dilation of the spaceship's unit time?

Solution: (i) Taking the spaceship's frame as the *S* one, the length in the frame *S*

$$L = L_0 \sqrt{1-\beta^2} \qquad \beta = v/c$$

It is given that $L = L_0/2$. Then

is given by

$$\frac{L_0}{2} = L_0\sqrt{1-\frac{v^2}{c^2}} \qquad \text{or} \qquad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$v = 0.866\,c$$

(ii) From Eq. (10.33), we have

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}} = \frac{\Delta\tau}{1/2} = 2\Delta\tau$$

That is, unit time in the *S* clock is recorded as twice of unit time by the observer. In other words, the spaceship's clock runs half as fast.

Example 10.4 An inertial frame *S* moves with respect to another inertial frame *S* with a uniform velocity 0.6 *c* along the *x x*-axes. The origins of the two systems coincide when $t = t = 0$. An event occurs at $x_1 = 10$ m, $y_1 = 0$, $z_1 = 0$, t_1 =

2×10^{-7}s. Another event occurs at $x_2 = 40$m, $y_2 = 0$, $z_2 = 0$, t_2 =

3×10^{-7}s . In *S*, (i) what is the time difference? (ii) what is the distance between the events?

Solution: (i) From Eq. (10.34 a)

$$t_2' - t_1' = \gamma(t_2 - t_1) + \frac{\gamma v}{c^2}(x_1 - x_2)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0.36}} = \frac{1}{\sqrt{0.64}} = 1.25$$

$$t_2' - t_1' = 1.25 \times (3-2)10^{-7}\,\text{s} + \frac{1.25 \times 0.6\,(-30\text{m})}{3 \times 10^8\,\text{ms}^{-1}}$$

$$= 1.25 \times 10^{-7}\,\text{s} - 0.75 \times 10^{-7}\text{s} = 0.5 \times 10^{-7}\text{s}$$

(ii) From Lorentz transformation $x_1' = \gamma(x_1 - vt_1) \qquad x_2' = \gamma(x_2 - vt_2)$

$$x_2' - x_1' = \gamma(x_2 - x_1) - \gamma v(t_2 - t_1)$$

$$= 1.25 \times 30\,\text{m} - 1.25(0.6)\left(3 \times 10^8\,\text{ms}^{-1}\right)10^{-7}\,\text{s}$$
$$= 37.5\,\text{m} - 22.5\,\text{m} = 15\,\text{m}$$

Example 10.5 How fast must an unstable particle. move to travel 20 m before it decays? The mean lifetime of the particle at rest = 2.6 × 10^{-8}s.

Solution: The mean lifetime of 2.6 10^{-8}s is in a frame of reference in which the particle is at rest. That is, *Dt* = 2.6 10^{-8}s. Lifetime in the laboratory frame

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} = \frac{2.6 \times 10^{-8}\,\text{s}}{\sqrt{1 - v^2/c^2}}$$

The distance travelled in the laboratory frame in time Δt is 20 m. Hence,

$$v = \frac{20}{\Delta t} \qquad \text{or} \qquad \Delta t = \frac{20}{v}\,\text{s}$$

Equating the two values for Δt

$$\frac{20}{v}\,\text{s} = \frac{2.6 \times 10^{-8}}{\sqrt{1 - v^2/c^2}}\,s$$

Squaring and simplifying

$$v = 2.8 \times 10^8\,\text{m/s}$$

Example 10.6 The average lifetime of *m*-mesons at rest is 2.3 10^{-6}s. A laboratory measurement on *m*-meson gives an average lifetime of 6.9 10^{-6}s. (i) What is the speed of the mesons in the laboratory ? (ii) What is the effective mass of a *m*-meson when moving at this speed, if its rest mass is $207m_e$?

(iii) What is its kinetic energy?

Solution: (i) Proper time interval *Dt* = 2.3 × 10^{-6}s. For the lifetime in the laboratory *Dt*, we have

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}} \quad \text{or} \quad 6.9 \times 10^{-6}\,\text{s} = \frac{2.3 \times 10^{-6}\,s}{\sqrt{1 - v^2/c^2}}$$

Solving, v = 0.9428c

(ii) $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \text{Since } \sqrt{1 - v^2/c^2} = \frac{2.3}{6.9},$$

$$m = 207 m_e \times \frac{6.9}{2.3} = 621 m_e$$

(iii) Kinetic energy $$T = mc^2 - m_0c^2 = (621 - 207)m_ec^2$$

$$T = 414(9.1 \times 10^{-31}\text{kg})\,(3 \times 10^8\,\text{ms}^{-1})^2 = 339.07 \times 10^{-13}\,\text{J}$$

$$= \frac{339.07 \times 10^{-13}\,\text{J}}{1.6 \times 10^{-19}\,\text{J/eV}} = 211.9 \times 10^6\,\text{eV} = 211.9\text{MeV}$$

Example 10.7 A *p*∞ meson of rest mass m_0, velocity *u* decays in flight into two photons of same energy. If one of the photons is emitted at an angle *q* to the direction of motion of the *p*∞ meson in the laboratory system, show that its energy *hv* is given by $$hv = \frac{m_0c^2}{2\gamma(1 - u\cos\theta/c)} \qquad \gamma = 1\Big/\sqrt{1 - u^2/c^2}$$

Solution: The two photons have the same energy. Hence, both will be making the same angle with the incident direction (See Fig. 10.10).

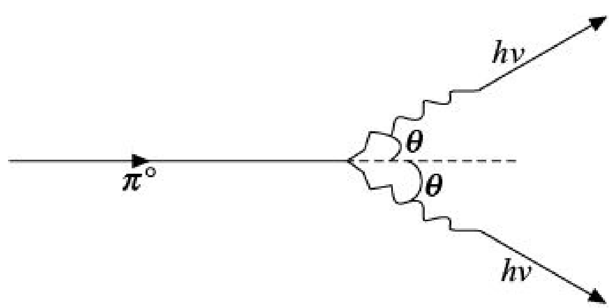


Fig. 10.10 *p*∞ meson disintegrating into two photons.

By the principle of conservation of energy

$$\gamma m_0 c^2 = 2h\nu \quad \text{(i)}$$

Conservation of momentum gives,

$$\gamma m_0 u = 2\frac{h\nu}{c}\cos\theta \quad \text{(ii)}$$

From Eqs. (i) and (ii)

$$\gamma m_0 u = \gamma m_0 c \cos\theta$$

or,

$$\cos\theta = \frac{u}{c} \quad \text{(iii)}$$

From Eq. (i)

$$h\nu = \frac{\gamma m_0 c^2}{2} = \frac{\gamma^2 m_0 c^2}{2\gamma}$$

$$= \frac{m_0 c^2}{2\gamma\left(1 - \frac{u^2}{c^2}\right)}$$

$$= \frac{m_0 c^2}{2\gamma(1 - u\cos\theta/c)}$$

Example 10.8 Energy and momentum conservation in pair production by photon is not possible if the process takes place in vacuum spontaneously. Prove.

Solution: Figure 10.11 illustrates pair production by a photon of energy *hv*.

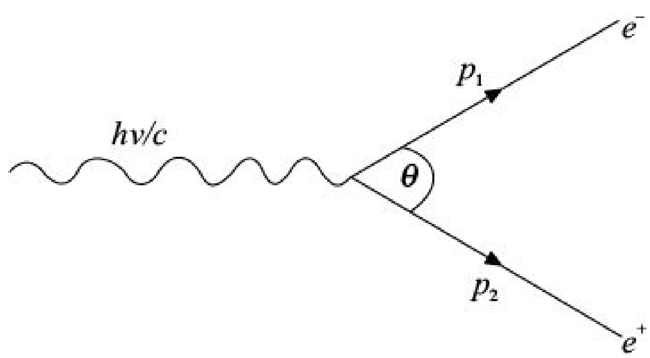


Fig. 10.11 Pair production by a photon.

Let p_1 be the momentum of the electron and p_2 be that of positron. The rest mass of electron and positron are the same, say m_0. The resultant of p_1 and p_2 can be obtained by the parallelogram law of velocities which must be equal to *hv*/*c*:

$$\left(\frac{hv}{c}\right)^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\theta \quad \text{(i)}$$

By law of conservation of energy,

$$hv = (c^2p_1^2 + m_0^2c^4)^{1/2} + (c^2p_2^2 + m_0^2c^4)^{1/2} \quad \text{(ii)}$$

Squaring and rearranging

$$\left(\frac{hv}{c}\right)^2 = p_1^2 + p_2^2 + 2m_0^2c^2 + 2(p_1^2 + m_0^2c^2)^{1/2}(p_2^2 + m_0^2c^2)^{1/2} \quad \text{(iii)}$$

From Eqs. (i) and (iii)

$$p_1p_2\cos\theta - m_0^2c^2 = (p_1^2 + m_0^2c^2)^{1/2}(p_2^2 + m_0^2c^2)^{1/2}$$

Squaring both sides

$$p_1^2p_2^2\cos^2\theta - 2p_1p_2m_0^2c^2\cos\theta = p_1^2p_2^2 + (p_1^2 + p_2^2)m_0^2c^2$$

$$p_1^2p_2^2(\cos^2\theta - 1) = m_0^2c^2(p_1^2 + p_2^2 + 2p_1p_2\cos\theta) \quad \text{(iv)}$$

$$-p_1^2p_2^2\sin^2\theta = m_0^2c^2(p_1^2 + p_2^2 + 2p_1p_2\cos\theta) \quad \text{(v)}$$

For the minimum value of cos*q* = –1, the right side of Eq. (v) is $m_0^2c^2(p_1 - p_2)^2$, which is positive. However, the left side of (v) is always negative or zero. Hence both sides must vanish. The left side vanishes for *q* = 0 or *p*. The right hand side vanishes only when *q* = *p* and $p_1 = p_2$. When this condition is satisfied, from Equation (i) we have *v* = 0. That is, the photon is nonexistent. Hence, such a process cannot take place.

Example 10.9 A *p*-meson of rest mass m_p decays at rest into a muon of rest mass m_m and a neutrino of zero rest mass. Evaluate the energy of the neutrino. Also show that the kinetic energy of the muon $T_\mu = (m_\pi - m_\mu)^2\, c^2/2m_\pi$.

Solution: Let *p* be the momentum of the neutrino. Then its energy = *cp*. By the law of conservation of momentum, the momentum of muon is equal and opposite to the momentum of neutrino. By the law of conservation of energy Energy of *p*-meson = Energy of *m*-meson + Energy of neutrino

$$m_\pi c^2 = \sqrt{c^2 p^2 + m_\mu^2 c^4} + pc$$

$$m_\pi c^2 - pc = \sqrt{c^2 p^2 + m_\mu^2 c^4}$$

Squaring and rearranging

$$m_\pi^2 c^4 - m_\mu^2 c^4 = 2 m_\pi c^2 pc$$

$$\text{Energy of the neutrino } = pc = \frac{(m_\pi^2 - m_\mu^2)\ c^2}{2m_\pi}$$

Kinetic energy of μ-meson $T_\mu =$ Energy of μ-meson $- m_\mu c^2$

$$T_\mu = \sqrt{c^2 p^2 + m_\mu^2 c^4} - m_\mu c^2 = m_\pi c^2 - pc - m_\mu c^2$$

$$= m_\pi c^2 - \frac{(m_\pi^2 - m_\mu^2)c^2}{2m_\pi} - m_\mu c^2$$

$$= \frac{2m_\pi^2 c^2 - m_\pi^2 c^2 + m_\mu^2 c^2 - 2m_\pi m_\mu c^2}{2m_\pi} = \frac{m_\pi^2 c^2 + m_\mu^2 c^2 - 2m_\pi m_\mu c^2}{2m_\pi}$$

$$= \frac{(m_\pi - m_\mu)^2 c^2}{2m_\pi}$$

Example 10.10 Show that the operator $\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is invariant under Lorentz transformation.

Solution: To prove the above invariance, one should know the relation between partial differentiation with respect to one set of variables (*x*, *y*, *z*, *t*) and the corresponding partial differentiation with respect to the other set of variables (*x*, *y*, *z*, *t*).The variables are related by the Lorentz transformation:

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{i}$$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \tag{ii}$$

From Eq. (i) we have

$$\frac{\partial x'}{\partial x} = \gamma \qquad \frac{\partial y'}{\partial x} = 0 \qquad \frac{\partial z'}{\partial x} = 0 \qquad \frac{\partial t'}{\partial x} = -\frac{\gamma v}{c^2}$$

Consequently,

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right) \tag{iii}$$

In the same way

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \qquad \frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\ \frac{\partial}{\partial x'}\right) \tag{iv}$$

$$\frac{\partial^2}{\partial x^2} = \left(\gamma\frac{\partial}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial}{\partial t'}\right)\left(\gamma\frac{\partial}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial}{\partial t'}\right)$$

$$= \gamma^2\frac{\partial^2}{\partial x'^2} + \gamma^2\frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2} - \frac{2v}{c^2}\gamma^2\frac{\partial^2}{\partial x'\partial t'} \tag{v}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2} \qquad \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z'^2} \tag{vi}$$

$$\frac{\partial^2}{\partial t^2} = \gamma^2\frac{\partial^2}{\partial t'^2} + \gamma^2 v^2\frac{\partial^2}{\partial x'^2} - 2\gamma^2 v\frac{\partial^2}{\partial x'\partial t'} \tag{vii}$$

Using Eqs. (v), (vi) and (vii), we get

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \gamma^2\frac{\partial^2}{\partial x'^2} + \gamma^2\frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2}$$

$$- \frac{2v}{c^2}\gamma^2\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$$

$$- \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2} - \gamma^2\frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2} + \frac{2\gamma^2 v}{c^2}\frac{\partial^2}{\partial x'\partial t'}$$

$$= \gamma^2\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{\gamma^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial t'^2}$$

$$= \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

That is, $\Box^2$ is Lorentz-invariant.

Example 10.11 Obtain the transformations for the components of the momentum-energy four-vector.

Solution: From Eq. (10.88), the momentum-energy four-vector is given by

$$P_\mu = (P_1, P_2, P_3, P_4) = \left(\mathbf{p}, \frac{iE}{c}\right) = (\mathbf{p}, imc)$$

For a four-vector $\mathbf{A}$, from Eq. (10.76) one has in inertial frame S'

$$A_1' = \gamma(A_1 + i\beta A_4) \qquad A_2' = A_2, \qquad A_3' = A_3, \qquad A_4' = \gamma(A_4 - i\beta A_1)$$

where $\beta = \mathrm{v}/c, \gamma = \frac{1}{\sqrt{1-\beta^2}}$. Using this relation, the momentum-energy four-vector components in inertial frame S' are given by

$$P_1' = \gamma\left(P_1 + \frac{i\mathrm{v}}{c}P_4\right) = \gamma\left(p_x + \frac{i\mathrm{v}}{c}\frac{iE}{c}\right) = \gamma\left(p_x - \frac{\mathrm{v}E}{c^2}\right)$$

$$p_x' = \gamma\left(p_x - \frac{\mathrm{v}E}{c^2}\right) \qquad p_y' = p_y \qquad p_z' = p_z$$

$$P_4' = \gamma\left(P_4 - \frac{i\mathrm{v}}{c}P_1\right) = \gamma\left(\frac{iE}{c} - \frac{i\mathrm{v}}{c}p_x\right) = \frac{i}{c}\gamma(E - \mathrm{v}p_x)$$

$$= \frac{i}{c}E', \qquad E' = \gamma(E - \mathbf{v}p_x)$$

Example 10.12 Obtain the transformations from inertial frame *S* to inertial frame *S* for the components of the four-force.

Solution: For a four-vector A_m, from Eq. (10.76) one has in inertial frame *S*

$$A_1' = \gamma\left(A_1 + \frac{i\mathrm{v}}{c}A_4\right), \qquad A_2' = A_2, \qquad A_3' = A_3, \qquad A_4' = \gamma\left(A_4 - i\frac{\mathrm{v}}{c}A_1\right)$$

From Eq. (10.95), for the four-force one has

$$F_\mu = \left(\frac{\mathbf{f}}{\sqrt{1-\left(\frac{u^2}{c^2}\right)}},\ \frac{i}{c}\frac{\mathbf{f}\cdot\mathbf{u}}{\sqrt{1-\left(\frac{u^2}{c^2}\right)}}\right) \tag{i}$$

Now the corresponding four-force in inertial frame S' is

$$F_\mu' = \left(\frac{\mathbf{f}'}{\sqrt{1-u'^2/c^2}}, \frac{i}{c}\frac{\mathbf{f}'\cdot\mathbf{u}'}{\sqrt{1-u'^2/c^2}}\right) \tag{ii}$$

Now, for the first component

$$F_1' = \gamma\left(F_1 + \frac{i\text{v}}{c}F_4\right) \tag{iii}$$

That is,

$$\frac{f_x'}{\sqrt{1-u'^2/c^2}} = \gamma\left(\frac{f_x}{\sqrt{1-u^2/c^2}} + \frac{i\text{v}}{c}\frac{i}{c}\frac{\mathbf{f}\cdot\mathbf{u}}{\sqrt{1-u^2/c^2}}\right)$$

$$f_x' = \frac{\sqrt{1-u'^2/c^2}}{\sqrt{1-\text{v}^2/c^2}\sqrt{1-u^2/c^2}}\left[f_x - \frac{\text{v}}{c^2}(f_x u_x + f_y u_y + f_z u_z)\right]$$

Since the factor outside the square bracket is $1/(1-\text{v}u_x/c^2)$ (See Example 10.13)

$$f_x' = \left[f_x - \frac{\text{v}}{c^2}(f_x u_x + f_y u_y + f_z u_z)\right]\frac{1}{\left(1-\frac{\text{v}u_x}{c^2}\right)}$$

$$= \left[f_x\left(1-\frac{\text{v}u_x}{c^2}\right) - \frac{\text{v}}{c^2}(f_y u_y + f_z u_z)\right]\frac{1}{\left(1-\frac{\text{v}u_x}{c^2}\right)}$$

$$= f_x - \frac{\text{v}u_y f_y}{c^2-\text{v}u_x} - \frac{\text{v}u_z f_z}{c^2-\text{v}u_x} \tag{iv}$$

Since $F_2' = F_2$

$$\frac{f_y'}{\sqrt{1-u'^2/c^2}} = \frac{f_y}{\sqrt{1-u^2/c^2}} \qquad \text{or} \qquad f_y' = \frac{\sqrt{1-u'^2/c^2}}{\sqrt{1-u^2/c^2}}f_y$$

Since $$\frac{\sqrt{1-u'^2/c^2}}{\sqrt{1-u^2/c^2}} = \frac{\sqrt{1-\text{v}^2/c^2}}{1-\text{v}u_x/c^2}$$

$$f_y' = \frac{c^2\sqrt{1-\text{v}^2/c^2}}{c^2-\text{v}u_x}f_y \quad \text{and} \quad f_z{}' = \frac{c^2\sqrt{1-\text{v}^2/c^2}}{c^2-\text{v}u_x}f_z \tag{v}$$

From Eq. (i)

$$F_4' = \gamma\left(F_4 - \frac{i\text{v}}{c}F_1\right)$$

$$\frac{i}{c}\frac{\mathbf{f}'\cdot\mathbf{u}'}{\sqrt{1-u'^2/c^2}} = \gamma\left(\frac{i}{c}\frac{\mathbf{f}\cdot\mathbf{u}}{\sqrt{1-u^2/c^2}} - \frac{i}{c}\frac{\text{v}f_x}{\sqrt{1-u^2/c^2}}\right)$$

$$\mathbf{f}'\cdot\mathbf{u}' = \frac{\sqrt{1-u'^2/c^2}}{\sqrt{1-u^2/c^2}\sqrt{1-\text{v}^2/c^2}}(\mathbf{f}\cdot\mathbf{u} - \text{v}f_x)$$

$$\mathbf{f}' \cdot \mathbf{u}' = \frac{\left(\mathbf{f} \cdot \mathbf{u} - \mathrm{v} f_x\right)}{1 - \dfrac{\mathrm{v} u_x}{c^2}}$$

The fourth relation involves the power since **f.u** has the unit of power.

Example 10.13 A particle is moving with a velocity **u** in an inertial frame *S* and with velocity **u** in inertial frame *S* which is moving parallel to the *x*-axis with a velocity v relative to *S*. Show that

$$\frac{\sqrt{1 - u'^2/c^2}}{\sqrt{1 - u^2/c^2}\,\sqrt{1 - \mathrm{v}^2/c^2}} = \frac{1}{(1 - \mathrm{v}^2\, u_x/c^2)}$$

Solution: From Eq. (10.86) the four-velocity

$$V_\mu = \left(\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}} \right) \quad \text{(i)}$$

In the S' frame

$$V'_\mu = \left(\frac{\mathbf{u}'}{\sqrt{1-u'^2/c^2}}, \frac{ic}{\sqrt{1-u'^2/c^2}} \right) \quad \text{(ii)}$$

According to Eq. (10.76)

$$V'_4 = \gamma\left(V_4 - i\frac{\text{v}}{c}V_1\right) = \gamma\left(\frac{ic}{\sqrt{1-u^2/c^2}} - \frac{i\text{v}}{c}\frac{u_x}{\sqrt{1-u^2/c^2}} \right)$$

$$= \gamma\, ic \frac{\left(1-\text{v}u_x/c^2\right)}{\sqrt{1-u^2/c^2}} \quad \text{(iii)}$$

But from (ii)

$$V'_4 = \frac{ic}{\sqrt{1-u'^2/c^2}} \quad \text{(iv)}$$

Equating the two expressions for V'_4

$$\frac{1}{\sqrt{1-u'^2/c^2}} = \frac{\gamma\left(1-\text{v}u_x/c^2\right)}{\sqrt{1-u^2/c^2}} \quad \text{(v)}$$

$$\frac{\sqrt{1-u'^2/c^2}}{\sqrt{1-u^2/c^2}\,\sqrt{1-\text{v}^2/c^2}} = \frac{1}{\left(1-\text{v}u_x/c^2\right)}$$

REVIEW QUESTIONS

1. What are inertial and noninertial frames of reference? Give examples.
2. Explain the significance of the null result of the Michelson-Morley experiment.
3. What is a Galilean transformation?
4. State the postulates of the special theory of relativity.
5. State the Lorentz transformation equations and express them in matrix form.

6. State and explain the relativistic law of addition of velocities.
7. Explain time dilation and length contraction.
8. Explain how the length contraction, time dilation and mass variation expressions might be used to indicate that c is the limiting speed in the universe.
9. What are proper time and proper length?
10. Show that the addition of a velocity to the velocity of light gives the velocity of light.
11. State the expressions for rest energy, kinetic energy and total energy of a relativistic particle.
12. Suppose the speed of light were infinite, what would happen to the relativistic predictions of length contraction, time dilation and mass variation?
13. ‘In special theory of relativity, mass and energy are equivalent.’ Discuss this statement with examples.
14. Will two events that occur at the same place and same time in one reference frame be simultaneous to an observer in a reference frame moving with respect to the first?
15. Does time dilation mean that time actually passes more slowly in moving reference frames or that it only seems to pass more slowly?
16. Does $E = mc^2$ apply to particles that travel at the speed of light?
17. Explain how relativity changed our notion about space and time.
18. ‘Events that are simultaneous in one reference frame are not simultaneous in another reference frame moving with respect to the first.’ Comment.
19. Is mass a conserved quantity in special relativity?
20. Show that the velocity of a particle having zero rest mass is equal to the velocity of light.
21. Draw the graph of kinetic energy versus momentum for (i) a particle of zero rest mass; (ii) a particle of nonzero rest mass.
22. What is a four-space? What is a world line?
23. When do you say an interval between two events is (i) time-like (ii) space-like (iii) light-like?
24. Comment on the statement: ‘The Lorentz transformation can be regarded as a rotation of coordinate axes x, y, z, ict in space time.’
25. Explain how the momentum components p_x, p_y, p_z along with iE/c, where E is the total energy, form a four-vector.
26. What is the charge-current four-vector? Express charge-current equation of continuity in electrodynamics in the covariant form.

27. What is a four-vector potential? Express Maxwell’s field equations in the four-vector form.
28. Explain the principle of equivalence. What is a geodesic?
29. What is gravitational red shift? Account for it.
30. Write notes on (i) precession of the perihelion of planetary orbits; (ii) bending of light in the gravitational field.

PROBLEMS

1. A rocket travelling away from the earth with a speed of 0.5 *c* fires off a second rocket at a speed of 0.6 *c* with respect to the first one. Calculate the speed of the second rocket with respect to the earth.
2. An object passes at a speed of 0.8 *c*. Its length is measured to be 72.5 m. At rest what would be its length?
3. At what speed would the relativistic value for time differ from the classical value by 2 per cent?
4. A person on a rocket travelling at a speed of 0.5 *c* with respect to the earth observes a meteor come from behind and pass him at a speed 0.5 *c*. How fast is the meteor moving with respect to the earth?
5. Two spaceships leave the earth in opposite directions, each with a speed of 0.5 *c* with respect to the earth . What is the velocity of spaceship 1 relative to spaceship 2?
6. A free neutron has an average lifetime of 1000 s. How fast must a beam of neutrons travel for them to have a lifetime twice this long with respect to the laboratory?
7. A proton has a kinetic energy of m_0c^2. Find its momentum in units of MeV/*c*.
8. A particle with mean lifetime of 10^{-6}s moves through the laboratory at a speed of 0.8 *c*. What will be its lifetime as measured by an observer in the laboratory?
9. What is the speed of a beam of particles if their mean lifetime is $3 \;\; 10^{-7}$s? Their proper lifetime is $2.6 \;\; 10^{-7}$s.
10. Calculate the rest energy in MeV of electron and proton. Mass of electron = $9.11 \;\; 10^{-31}$ kg, Mass of proton = $1.67 \;\; 10^{-27}$kg.
11. What is the kinetic energy of a proton moving at a speed of 0.86 *c*? Its rest energy is 939 MeV.
12. If the kinetic energy of an electron is 5 MeV, what is its velocity?

13. A meson having a mass of 2.4 10^{-28} kg travels at a speed of **v** = 0.8 *c*. What is its kinetic energy?
14. At what speed will the mass of a body be twice its rest mass?
15. Evaluate the speed of a particle when its kinetic energy equals its rest energy.
16. Calculate the mass of a particle whose kinetic energy is half its total energy. How fast is it travelling?
17. What is the speed and momentum of an electron whose kinetic energy equals its rest energy?
18. The mean lifetime of a muon at rest is 2.4 10^{-6}s. What will be its mean lifetime as measured in the laboratory, if it is travelling at **v** = 0.6 *c* with respect to the laboratory?
19. p-mesons coming out of an accelerator have a velocity of 0.99 *c*. If they have a mean lifetime of 2.6 10^{-8}s in the rest frame, how far can they travel before decay?
20. At what speed will the relativistic value for length differ from the classical value by 1 per cent?
21. A beam of particles travels at a speed of 0.9 *c*. At this speed the mean lifetime as measured in the laboratory frame is 5 10^{-6}s. What is the particle's proper lifetime?
22. At what speed will the mass of a body be 20 per cent greater than its rest mass?
23. Derive an expression showing how the density of an object changes with speed **v** relative to an observer.
24. If the sun radiates energy at the rate of 4 10^{26}Js^{-1}, evaluate the rate at which its mass is decreasing.
25. Find the momentum and velocity of an electron having kinetic energy of 10.0 MeV. The rest energy of electron is 0.512 MeV.
26. A particle having rest mass m_0 is travelling at a speed of u. If its momentum is p, show that (i) $p = m_0 c\left[\frac{1}{1-u^2/c^2} - 1\right]^{1/2}$ (ii) $\frac{1}{1-u^2/c^2} = 1 + \frac{p^2}{m_0^2 c^2}$
27. An electron is accelerated to an energy of 1.0 BeV. (i) What is its effective mass in terms of its rest mass. (ii) What is its speed in terms of the speed of light?
28. Derive the following relations between momentum p, kinetic energy T and

rest mass m_0 for relativistic particles:

(i) $T = c\sqrt{m_0^2c^2 + p^2} - m_0c^2$ (ii) $p = \frac{\sqrt{T^2 + 2m_0c^2T}}{c}$

29. Compute the effective mass for a photon of wavelength 5000 Å and for a photon of wavelength 1.0 Å.

30. Show that the rest mass of a particle of kinetic energy *T* and momentum *p* is given by $m_0 = \frac{c^2p^2 - T^2}{2c^2T}$.

31. Show that (i) a particle which travels at the speed of light must have a zero rest mass; (ii) for a particle of zero rest mass, $T = E$, $p = E/c$.

32. A charged *p*-meson of rest mass 273 m_e at rest decays into a neutrino and a *m*-meson of rest mass 207 m_e. Find the kinetic energy of the *m*-meson and the energy of the neutrino.

33. Two electrons are ejected in opposite directions from a radioactive nucleus which is at rest in a laboratory. If each electron has a speed of 0.67 *c* as measured by a laboratory observer, what is the speed of one electron relative to the other?

34. A p^+-meson is created in the earth's atmosphere 200 km above the sea level. It descends vertically at a speed of 0.99 *c* and disintegrates in its proper frame 2.5 10^{-8} s after its creation. At what altitude above its sea level is it observed to disintegrate?

11

Introduction to Nonlinear Dynamics

The mechanical problems we considered so far have linear time evolution equation. However, most of the dynamical systems and phenomena in nature are nonlinear. We are not fully equipped with simple tools to handle nonlinear problems, although linear ones have been extensively studied. Many of the nonlinear problems are reduced to linear ones by approximations. In many situations such linearization procedures are valid to a large extent. In this context a remark by Albert Einstein is worth noting : 'Since the basic equations of physics are nonlinear, all of mathematical physics will have to be done again.' In this chapter, we shall discuss some of the general features of nonlinear dynamic systems, time-dependence and stability of their solutions.

11.1 LINEAR AND NONLINEAR SYSTEMS

In Example 1.6, we discussed the familiar linear harmonic oscillator having frequency $\omega_0 = \sqrt{k/m}$, where *k* is the spring constant and *m* is the mass of the particle executing the motion. The force acting on the system

$$F_x = m\frac{d^2x}{dt^2} = -kx \quad (11.1)$$

The time evolution equation for the position of the particle is then

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \quad (11.2)$$

This equation is linear in x and in the second derivative of x. Its general solution is

$$x = A\sin(\omega_0 t + \phi) \quad (11.3)$$

where the amplitude A and the phase *f* are constants. That is, if the mass is displaced from the equilibrium position, it will oscillate sinusoidally about that position with an angular frequency w_0. In this example, we have a **linear system.**

Next we shall consider a slightly more complicated system. If the system has an additional force term of the type bx^2, the time evolution of the equation becomes

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \frac{b}{m}x^2 = 0 \quad (11.4)$$

Now the system is a nonlinear one, since the position *x* of the particle in the equation is a squared one.

A system whose time evolution equations are nonlinear is called a ***nonlinear system****. The dynamical variables describing the properties of the variables such as position, velocity, acceleration, etc. appear in the equations in a nonlinear form.*

In linear systems, if $f_1(x, t)$ and $f_2(x, t)$ are linearly independent solutions of the time evolution equation for the system, then the linear combination $c_1 f_1(x, t) + c_2 f_2(x, t)$ where c_1 and c_2 are constants, is also a solution. However, it is not in the nonlinear case. We can also explain the concept of nonlinearity in terms of the response of a system to a stimulus. Let a stimulus $s(t)$ give rise to a response $g(x, t)$ to a particular system. If we change the stimulus to 2 $s(t)$, a linear system will have the response 2 $g(x,t)$. For a nonlinear system, the response will be different from 2 $g(x, t)$, can be smaller or larger than 2 $g(x, t)$. That is, if a parameter that describes a linear system is changed, then the other parameters will change correspondingly. For nonlinear systems, a small change in a parameter can lead to dramatic and sudden changes of the co-ordinates and other parameters in both qualitative and quantitative behaviour of the system. It may be noted here that most of the real systems are nonlinear at least to some extent.

11.2 INTEGRATION OF LINEAR EQUATION: QUADRATURE METHOD

Again consider the linear equation, Eq. (11.2), for our discussion. It can be written in the form of a pair of coupled first order equations:

$$\dot{x} = y \tag{11.5a}$$

$$\dot{y} = -\,\omega_0^2 x \tag{11.5b}$$

Multiplying Eq. (11.5a) by $\omega_0^2 x$ and Eq. (11.5b) by y and adding the two

$$y\dot{y} + \omega_0^2 x\dot{x} = 0 \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{1}{2}y^2 + \frac{1}{2}\omega_0^2 x^2\right) = 0$$

Integrating

$$\frac{1}{2}\left(y^2 + \omega_0^2 x^2\right) = I_1 \tag{11.6}$$

The constant I_1 is called the *first integral*. Since $y = \dot{x}$, therefore $y^2/2$ may be identified with scaled kinetic energy and $\omega_0^2 x^2/2$ with potential energy. Therefore, I_1 can be identified as the scaled total energy E'. From Eq. (11.6)

$$y = \frac{dx}{dt} = \sqrt{2I_1 - \omega_0^2 x^2}$$

$$dt = \frac{dx}{\sqrt{2I_1 - \omega_0^2 x^2}} \tag{11.7}$$

This equation can now be solved as an explicit integral or **quadrature**. That is,

$$\int dt = \int \frac{dx}{\sqrt{2I_1 - \omega_0^2 x^2}} \tag{11.8}$$

After integrating both sides, we have a second constant of integration I_2 and therefore we can write

$$t + I_2 = \int \frac{dx}{\sqrt{2I_1 - \omega_0^2 x^2}} = \int \frac{dx}{\sqrt{2I_1(1 - \omega_0^2 x^2/2I_1)}} \tag{11.9}$$

Evaluation of the integral is straightforward if the substitution $\sin\theta = \omega_0 x/\sqrt{2I_1}$ is done. Integration gives

$$t + I_2 = \frac{1}{\omega_0} \sin^{-1}\left(\frac{\omega_0 x}{\sqrt{2I_1}}\right) \tag{11.10}$$

$$x(t) = \frac{\sqrt{2I_1}}{\omega_0} \sin(\omega_0 t + I_2 \omega_0) \tag{11.11}$$

which is the same as Eq. (11.2). Though this procedure is somewhat roundabout for this simple linear equation, it becomes more natural when nonlinear terms appear in the differential equation. Even the method of quadrature fails in certain cases where the nonlinearity is higher than second order.

It is obvious from Eq. (11.11) that the period of the oscillator $T = 2\pi/\omega_0$. This can also be obtained from Eq. (11.8). Since I_1 is the total energy E and at the classical turning points $y = \dot{x} = 0$, from Eq. (11.6)

$$\omega_0^2 x^2 = 2E' \qquad \text{or} \qquad x = \pm\sqrt{2E'}/\omega_0$$

Hence, the period of the pendulum T, from Eq. (11.8), is

$$T = 2\int_{-\sqrt{2E'}/\omega_0}^{\sqrt{2E'}/\omega_0} \frac{dx}{\sqrt{2E' - \omega_0^2 x^2}} = \frac{2\pi}{\omega_0} \tag{11.12}$$

Thus, the period is independent of energy.

11.3 INTEGRATION OF NONLINEAR SECOND ORDER EQUATIONS

Nonlinear second order equations are very common in physics and a majority of them are in the form $\dfrac{d^2x}{dt^2} + f(x) = 0$ (11.13)

where $f(x)$ might be a polynomial, rational or transcendental functions of x. As an example, we may consider a particle moving under a force function which is of third order in the displacement x. Then the equation of motion is

$$\frac{d^2x}{dt^2} = -f(x) = -(A + Bx + Cx^2 + Dx^3) \tag{11.14}$$

where A, B, C, D are constants. Equation (11.13) can be put in the form

$$\dot{x} = y \qquad \dot{y} = -f(x) \tag{11.15}$$

Dividing one equation by the other

$$\frac{\dot{y}}{\dot{x}} = \frac{-f(x)}{y} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{-f(x)}{y}$$

$$y\,dy = -f(x)\,(dx) \tag{11.16}$$

Integrating

$$\frac{1}{2}y^2 = -\int f(x)dx + \text{constant}$$

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \int f(x)dx = I_1 \text{ (constant)} \tag{11.17}$$

Replacing $f(x)$ using Eq. (11.14) and performing the integration

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \left(Ax + \frac{B}{2}x^2 + \frac{C}{3}x^3 + \frac{D}{4}x^4\right) = I_1 \tag{11.18}$$

For convenience, we may rewrite Eq. (11.18) in the form

$$\left(\frac{dx}{dt}\right)^2 = a + bx + cx^2 + dx^3 + ex^4 \tag{11.19}$$

The right hand side of Eq. (11.19) can be factored as

$$e(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

Consequently, Eq. (11.19) can be written as

$$\left(\frac{dx}{dt}\right)^2 = e(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \tag{11.20}$$

which can be transformed into the standard Legendre form

$$\left(\frac{dx}{dt}\right)^2 = (1-x^2)(1-k^2x^2) \tag{11.21}$$

$$dt = \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{11.22}$$

Integrating

$$\int dt = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{11.23}$$

The integral on the right hand side of Eq. (11.23), denoted as $F(x, k)$, is called the **elliptic integral** of the first kind.

$$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{11.24}$$

An alternate form of Eq. (11.24) can easily be obtained by making the substitution $x = \sin\,\theta$.

$$F(\theta,k) = \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin\theta}} \tag{11.25}$$

Tabulated values of the elliptic integral are available in mathematical handbooks. The inverse of the elliptic integral given in Eq. (11.25) or Eq. (11.24) are the **Jacobi's elliptic functions**. For more details regarding elliptic integrals, refer to Appendix A.

11.4 THE PENDULUM EQUATION

Another most studied nonlinear system is the simple pendulum. It can be integrated exactly in terms of elliptic functions. Let the departure of the string from its equilibrium position be q. The time evolution equation for the position of the bob can be written as $$\frac{d^2\theta}{dt^2} + \frac{g \sin\theta}{l} = 0 \qquad (11.26)$$

where l is the length of the pendulum. If we approximate sin *q* by*q* we get the familiar equation of a harmonic oscillator

$$\ddot{\theta} + \omega_0^2\theta = 0 \qquad \omega_0^2 = \frac{g}{l} \qquad (11.27)$$

with the general solution given by Eq. (11.3).

We shall now solve the original nonlinear equation without the approximation. Writing Eq. (11.26) as two coupled equations

$$\dot{\theta} = y \qquad \text{and} \qquad \dot{y} = -\omega_0^2 \sin\theta \tag{11.28}$$

Dividing one equation by the other

$$\frac{dy}{d\theta} = -\frac{\omega_0^2 \sin\theta}{y}$$

$$y\,dy = -\omega_0^2 \sin\theta\, d\theta \tag{11.29}$$

Integrating

$$\frac{1}{2}y^2 = \omega_0^2 \cos\theta + \text{constant} \tag{11.30}$$

The constant can easily be identified as scaled mechanical energy E', which is equal to E/ml^2, where E is the total energy. From Eq. (11.30)

$$\frac{d\theta}{dt} = \sqrt{2(E' + \omega_0^2 \cos\theta)} \qquad \text{or} \qquad dt = \frac{d\theta}{\sqrt{2(E' + \omega_0^2 \cos\theta)}}$$

Integrating

$$t = \int \frac{d\theta}{\sqrt{2(E' + \omega_0^2 \cos\theta)}} + \text{constant} \tag{11.31}$$

The constant of integration may be dropped as it produces only a phase shift. To change the integral in Eq. (11.31) into a standard form, let us introduce the new parameter ω by the substitution

$$\cos\omega = -\frac{E'}{\omega_0^2} \tag{11.32}$$

In terms of ω, Eq. (11.31) takes the form

$$t = \sqrt{\frac{l}{2g}} \int_0^{\theta} \frac{d\theta'}{\sqrt{\cos\theta' - \cos\omega}} \tag{11.33}$$

For further simplification introduce the transformation

$$\cos\theta = 1 - 2k^2 \sin^2\phi \qquad \text{and} \qquad k = \sin\frac{\omega}{2} \tag{11.34}$$

Since $\qquad \cos\omega = 1 - 2\sin^2(\omega/2) = 1 - 2k^2$

$$\cos\theta - \cos\omega = 1 - 2k^2 \sin^2\phi - (1 - 2k^2) = 2k^2(1 - \sin^2\phi) = 2k^2 \cos^2\phi$$

$$\frac{1}{\sqrt{\cos\theta - \cos\omega}} = \frac{1}{\sqrt{2k^2 \cos^2\phi}} = \frac{1}{\sqrt{2}k \cos\phi} \qquad (11.35)$$

The definition of *k* suggests that it is a constant. Hence, differentiating Eq. (11.34)

$$-\sin\theta\, d\theta = -\,2k^2 \times 2\sin\phi\cos\phi\, d\phi$$

$$d\theta = \frac{2k^2 \sin\phi\cos\phi\, d\phi}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \tag{11.36}$$

From Eq. (11.34)

$$2k^2\sin^2\phi = 1-\cos\theta = 1-\left(1-2\sin^2\frac{\theta}{2}\right) = 2\sin^2\frac{\theta}{2}$$

Hence,

$$\sin\frac{\theta}{2} = k\sin\phi \qquad \text{and} \qquad \cos\frac{\theta}{2} = \sqrt{1-k^2\sin^2\phi} \tag{11.37}$$

From Eqs. (11.35) to (11.37)

$$\frac{d\theta}{\sqrt{\cos\theta-\cos\omega}} = \frac{2k^2\sin\phi\cos\phi\, d\phi}{\sqrt{2}\,k\cos\phi\times k\sin\phi\sqrt{1-k^2\sin^2\phi}}$$

$$= \frac{\sqrt{2}\, d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{11.38}$$

Consequently Eq. (11.33) reduces to

$$t = \sqrt{\frac{l}{g}}\int_0^{\phi}\frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}} \tag{11.39}$$

where,

$$k = \sin\frac{\omega}{2} = \sqrt{\frac{1}{2}\left(1+\frac{E'}{\omega_0^2}\right)} \tag{11.40}$$

In terms of elliptic functions

$$sn\left(t\sqrt{\frac{g}{l}}, k\right) = \sin\phi = \frac{1}{k}\sin\frac{\theta}{2}$$

Angle $q(t)$ in terms of time can be written as

$$\sin\frac{\theta(t)}{2} = k\ sn\left(t\sqrt{\frac{g}{l}}, k\right)$$

$$\theta(t) = 2\sin^{-1}\left[k\ sn\left(t\sqrt{\frac{g}{l}}, k\right)\right] \qquad (11.41)$$

11.5 PHASE PLANE ANALYSIS OF DYNAMICAL SYSTEMS

For understanding the dynamics of linear and nonlinear systems, the description of its behaviour in phase space is quite useful. The two independent variables (x, p_x) here $(x, y = \dot{x})$, define the space in which the solution moves. For a particle having only one independent variable, the phase space is only two-dimensional and hence it is often referred to as the **phase plane**. At any time the value of the phase space co-ordinates (x, y) completely defines the state of the system. For a system having n independent variables $x_1, x_2, \ldots, x_n$, each variable can be thought of as an independent phase space co-ordinate in the associated n-dimensional phase space. A given solution to the equations of motion will map out a smooth curve in the phase plane as a function of time. This is called a **phase curve** or **phase trajectory** and the motion along it is called the **phase flow**. Because of the unique properties of solutions to differential equations, different phase space trajectories do not cross each other. A picture made up of sets of phase curves is often called a **phase portrait**.

Phase Curve of Simple Harmonic Oscillator To illustrate the various concepts, we make use of the familiar harmonic oscillator problem as given in Eq. (11.2). In its first integral, Eq. (11.6), I_1 is simply total energy. From Eq. (11.6)

$$\frac{y^2}{2E} + \frac{\omega_0^2}{2E}x^2 = 1 \qquad (11.42)$$

Clearly the phase trajectories are concentric ellipses (See Fig. 11.1). The semi-major and minor axes can easily be determined as detailed below. At the turning points of the ellipse, $y = \dot{x} = 0$. Hence, from Eq. (11.42)

$$\omega_0^2 x^2 = 2E \qquad \text{or} \qquad x = \pm\sqrt{2E}/\omega_0$$

When $x = 0$, again from Eq. (10.42)

$$y^2 = 2E \qquad \text{or} \qquad y = \pm\sqrt{2E}$$

Hence, the semi-major axis a and semi-minor axis b are given by

$$a = \frac{\sqrt{2E}}{\omega_0} \qquad b = \sqrt{2E} \tag{11.43}$$

The origin of the phase plane $x = y = 0$ corresponds to an obvious equilibrium point of the motion. Thus, the existence of a constant first integral has provided a definite geometrical constraint on the phase flow.

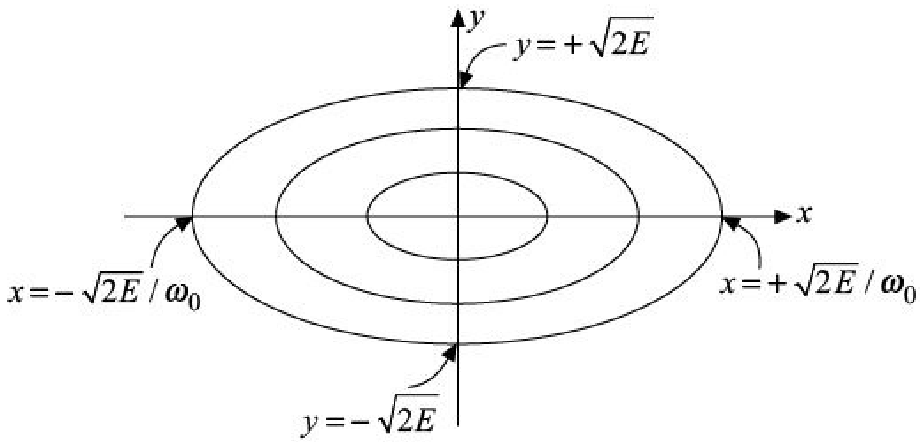


Fig. 11.1 Phase curves for the simple harmonic oscillator.

Phase Curve of Damped Oscillator

Next we shall consider oscillations with damping forces proportional to velocity. The equation of the oscillator is then

$$\ddot{x} + 2b\,\dot{x} + \omega_0^2 x = 0 \tag{11.44}$$

This is equivalent to the coupled equations

$$y = \dot{x} \qquad \text{and} \qquad \dot{y} = -\,\omega_0^2 x - 2by \tag{11.45}$$

The trial solution

$$x = e^{\lambda t} \tag{11.45a}$$

in Eq. (11.44) gives the equation

$$e^{\lambda t}\left(\lambda^2 + 2b\lambda + \omega_0^2\right) = 0 \quad \text{or} \quad \lambda^2 + 2b\lambda + \omega_0^2 = 0$$

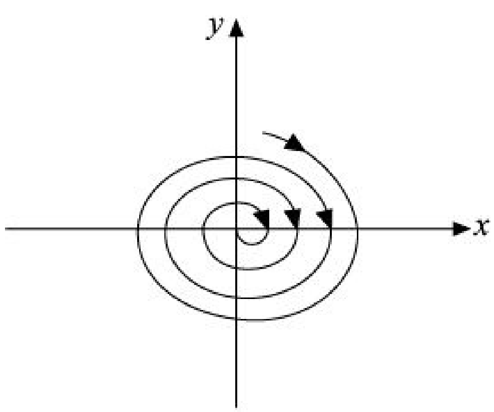
The quadratic in λ gives the roots

$$\lambda_1, \lambda_2 = -\,b \pm \sqrt{b^2 - \omega_0^2} \tag{11.46}$$

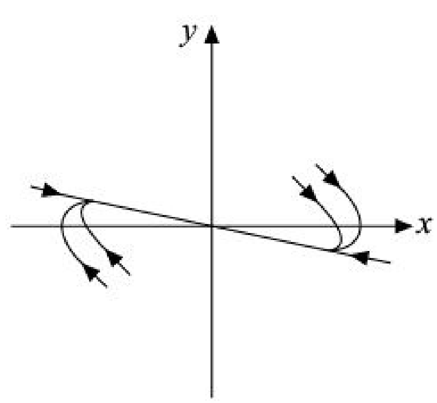
When $b^2 < \omega_0^2$, the roots l_1 and l_2 are complex and the motion is oscillation with decreasing amplitude. The solution just spirals into the equilibrium point at the origin at a rate depending on the damping coefficient b (See Fig. 11.2a).

In the case of $b^2 > \omega_0^2$ or $b^2 = \omega_0^2$, the solution is aperiodic damped motion. Phase trajectories for these cases are obtained numerically and are given in Figs. 11.2 (b) and (c). The trajectories in these two cases approach the origin.

For nonlinear systems such as Eq. (11.13) with a constant first integral and executing bounded motion, the phase portrait is again a set of concentric curves centred at the origin. However, for more general nonlinear systems, the phase portrait is more complicated.



(a)



(b)

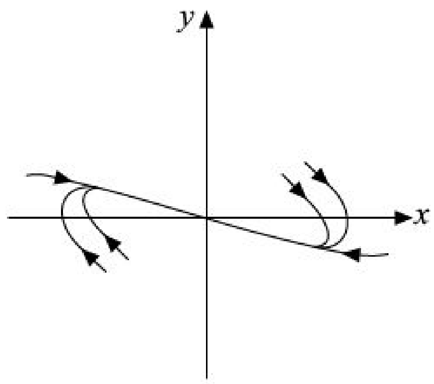


Fig. 11.2 Phase curves of a damped oscillator: (a) $b^2 < \omega_0^2$. The solution spirals into the equilibrium point; (b) $b^2 > \omega_0^2$; (c) $b^2 = \omega_0^2$. In (b) and (c) the trajectories approach the origin.

11.6 PHASE PORTRAIT OF THE PENDULUM

One of the most studied examples is the simple pendulum. Its time evolution equation for the displacement is given by Eq. (11.26). To have uniformity in notation with the previous section, we denote the variables θ and $\dot{\theta}$ as x and y, respectively. In the new notation Eq. (11.28) takes the form

$$\dot{x} = y \qquad \text{and} \qquad \dot{y} = -\omega_0^2 \sin x \qquad (11.47)$$

Proceeding as in Section 11.4, in place of Eq. (11.30), we get

$$\frac{1}{2}y^2 = \omega_0^2 \cos x + E' \qquad (11.48)$$

where E is the scaled total energy. The phase space diagram of the pendulum is shown in Fig. 11.3.

For very small energies the pendulum will just oscillate about the equilibrium point $x = y = 0$ in nearly linear fashion. For small energies the phase space trajectories are ellipses centred on the origin. As the energy increases, the pendulum

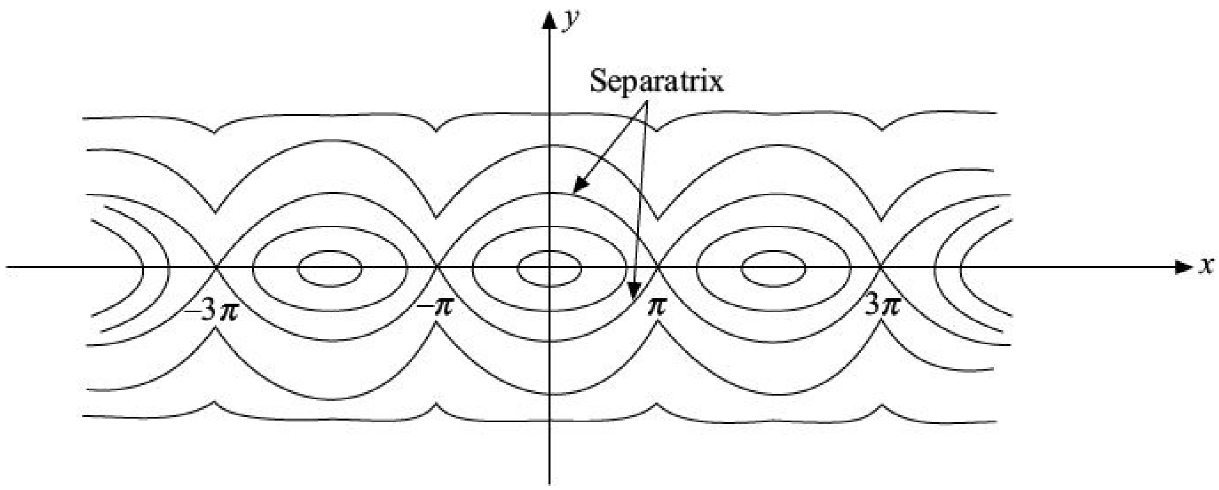


Fig. 11.3 Phase curves for the pendulum.

executes larger librations until finally a point is reached with the pendulum standing straight up with the mass directly above the point of pivot and starts to execute rotational motion. That means the pendulum has sufficient energy to swing from *x* = 0 to $x = \pm\pi$, the value of *y* for these two values of *x* being zero. When *y* = 0 and $x = \pm\pi,$ from Eq. (11.48) $E' = \omega_0^2 = g/l$. In other words, the pendulum will just complete the circle if it has the energy $E' = g/l$. As the energy increases further and further, the rotational motion gets faster and faster. The point $\dot{\omega} = \pm$ with *y* = 0 is an equilibrium point, but an unstable one. This phase pattern will be repeated at every multiple of 2*p* to the left and right since the restoring force is periodic. Thus, at every $x = \pm 2n\pi$ there is a **stable equilibrium** point and at every $x = \pm(2n+1)\pi$ there is an **unstable equilibrium** point. These points marks a transition from librational motion to rotational motion and the phase curves change from closed to open ones. The open one corresponds to unbounded rotational motion.

The pair of space curves that separate the librational and rotational motions and that meet at the unstable equilibrium points is termed the **separatrix**. Inside the separatrices, the motion is completely periodic and oscillatory. Trajectories outside the separatrices correspond to **running modes** in which the pendulum has sufficient energy to swing over the top. One type of running mode has angular velocity $\dot{e} > 0$ (anticlockwise motion) and the other type has angular velocity $\dot{e} < 0$ (clockwise motion).

11.7 MATCHING OF PHASE CURVE WITH POTENTIAL *V*(*x*)

For conservative systems, the energy *E* is a constant of motion and can be written as the sum of kinetic and potential energies. We may write $E = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}y^2 + V(x)$ (11.49) where the potential function is some nonlinear function of *x*. For conservative systems, the phase curves are often referred to as **level curves** and their constructions are somewhat easy.

Simple Harmonic Oscillator

For the simple harmonic oscillator $V(x) = \frac{1}{2}kx^2$ and the curve *V*(*x*) versus *x* is a simple parabola in which the motion is confined between the classical turning points given by $x = \pm\sqrt{2E}/\omega_0.$ Matching of this parabola with the phase curve of the oscillator (See Fig. 11.1) is illustrated in Fig. 11.4. Clearly the ellipses are between the classical turning points in the parabola.

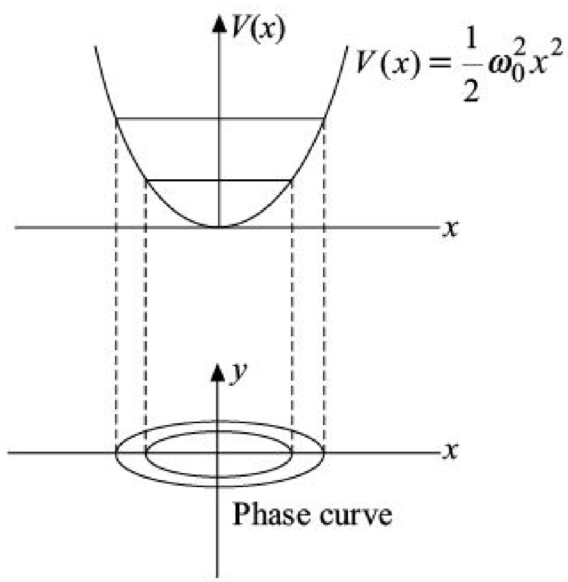


Fig 11.4 Matching of phase curves with potential *V*(*x*) for a simple harmonic oscillator.

Simple Pendulum

For the pendulum, $V(x) = -(g/l)\cos x,$ which is a nonlinear periodic potential (See Fig. 11.5). Below the periodically spaced maxima the motion is bounded and hence elliptical. Above the maxima, as discussed the motion is rotational and hence unbounded. The minima are stable equilibrium points whereas the maxima give the unstable equilibrium points. The matching of the potential energy curve with the phase curve is illustrated in Fig. 11.5.

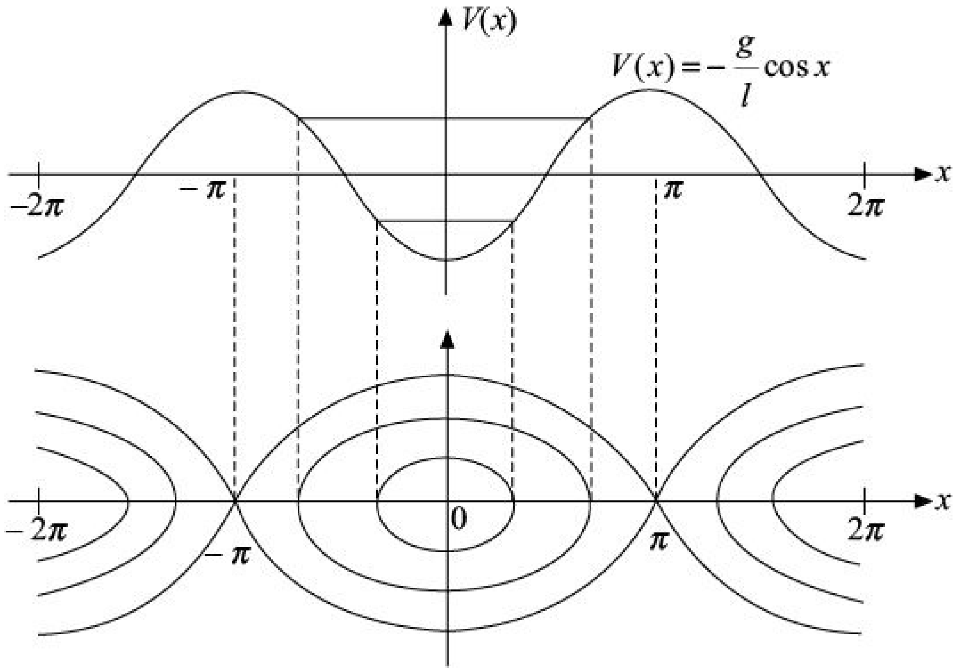


Fig. 11.5 Matching of phase curves with potential *V*(*x*) for a pendulum.

11.8 LINEAR STABILITY ANALYSIS

Phase portraits for conservative systems are not difficult to construct. They have characteristic closed curves around stable points and hyperbolic type regions in the neighbourhood of unstable points. For nonconservative systems, construction of phase portraits are difficult unless an explicit solution is known to the equations of motion. However, it is possible to construct an approximate local phase portrait by identifying the equilibrium points, referred to as **fixed points** or **critical points** and drawing phase curves in their neighbourhood. Fixed points will be satisfying conditions satisfied by equilibrium points. Therefore, one can build up a fairly good phase portrait of any system by identifying reasonably stable fixed points, which are the organizing centres of a system’s phase space dynamics.

Stability Matrix

For discussion, consider a general second order system of the form

$$\dot{x} = f(x, y) \tag{11.50a}$$

$$\dot{y} = g(x, y) \tag{11.50b}$$

where f and g are typical nonlinear smooth functions of x and y. Let the values of x and y at fixed points of the motion be x_0 and y_0, respectively. As $\dot{x} = \dot{y} = 0$ at fixed points

$$f(x_0, y_0) = 0 \tag{11.51a}$$

$$g(x_0, y_0) = 0 \tag{11.51b}$$

The number of such fixed points depend on the form of *f* and *g*. The stability of fixed points can be understood by considering the time evolution of some small displacement $(\delta x, \delta y)$ about (x_0, y_0). For small displacements, we need retain only the linear terms in the Taylor expansion of the functions *f* and *g*. The linearized time evolution of displacements *dx* and *dy* are then given by

$$\delta\dot{x} = f_x(x_0, y_0)\,\delta x + f_y(x_0, y_0)\delta y \tag{11.52a}$$

$$\delta\dot{y} = g_x(x_0, y_0)\,\delta x + g_y(x_0, y_0)\delta y \tag{11.52b}$$

Equations (11.52a) and (11.52b) can be expressed in the matrix form as

$$\frac{d}{dt}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0)\\ g_x(x_0, y_0) & g_y(x_0, y_0)\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} \tag{11.53}$$

Equation (11.53) is often referred to as the *linearized equations*. The 2 2 matrix in Equation (11.53), denoted as **M**, is often referred to as the **stability matrix**. Let l_1 and l_2 be the two eigenvalues of the matrix **M** and the eigenvectors associated with the eigenvalues be $\mathbf{D_1}$ and $\mathbf{D_2}$. A solution of the first order linear equations in Equation (11.53) is given by the roots of the equation

$$\det \left| \mathbf{M} - \lambda \mathbf{I} \right| = 0 \tag{11.54}$$

where $\boldsymbol{I}$ is the unit matrix. The general solution of Eq. (11.53) is given by

$$\delta \mathbf{X} = c_1 \mathbf{D}_1 e^{\lambda_1 t} + c_2 \mathbf{D}_2 e^{\lambda_2 t} \tag{11.55}$$

where $\delta \mathbf{X}$ stands for the column vector formed by δx and δy. This procedure can easily be extended to nth order system of the form

$$\dot{x} = f_i(x_1, x_2, \ldots, x_n), i = 1, 2, 3, \ldots, n \; .$$

Classification of Fixed Points The nature of phase curves will depend on the eigenvalues of the stability matrix l_1 and l_2. However, the form of eigenvectors determines the actual directions of the local phase flows. The different possibilities are discussed in this section.

Case (i): $l_1 < l_2 < 0$– a **stable node**. As the eigenvalues are negative, the local flow decays in both directions determined by $\mathbf{D}_1$ and $\mathbf{D}_2$ into the fixed point, as illustrated in Fig. 11.6 (a).

Case (ii): $l_1, l_2 > 0$–an **unstable node**. The local flow grows exponentially away from the fixed point in both directions, as shown in Fig. 11.6 (b).

Case (iii): $l_1 < 0 < l_2$–**hyperbolic point or saddle point**. One direction grows exponentially and the other decays exponentially, as illustrated in Fig. 11.6 (c). The incoming and outgoing directions are often referred to as the *stable* and *unstable manifolds* of the separatrix, respectively.

Case (iv): $\lambda_1 = -\alpha + i\beta, \lambda_2 = -\alpha - i\beta, (\alpha, \beta > 0)$—**a stable spiral point**. Since the two eigenvalues l1 and l2 have the negative real part –a, the flow spirals in toward the fixed points, as shown in Fig. 11.6 (d).

Case (v): $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$— an **unstable spiral point**. Because of the positive real parts, the flow spirals away from the fixed point (Fig. 11.6 e).

Case (vi): $\lambda_1 = i\omega, \lambda_2 = -i\omega$—an **elliptic point** or simply **centre**. As the two eigenvalues are purely imaginary, the phase curves will be closed ellipses, as shown in Fig 11.6 (f). This will be a stable equilibrium point.

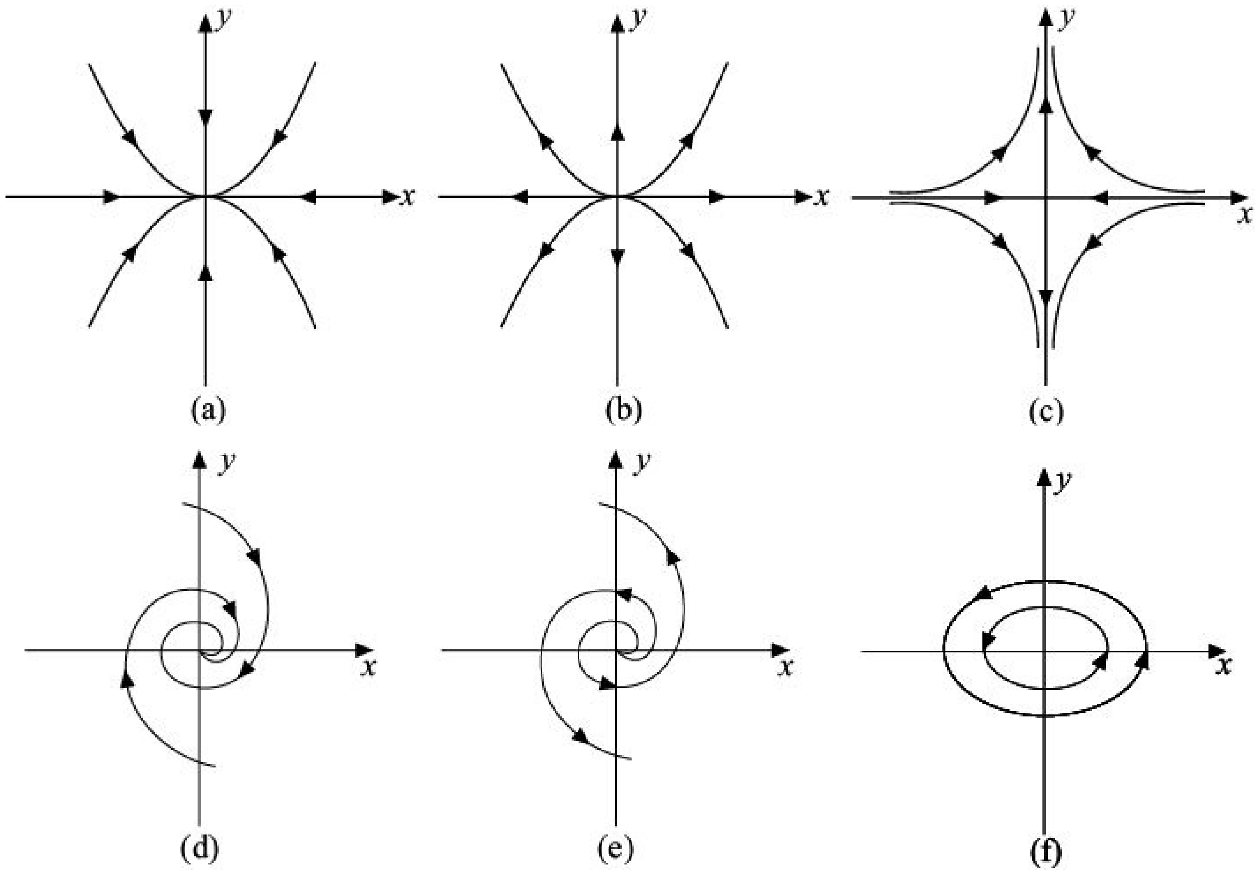


Fig. 11.6 Local phase flows for: (a) stable node; (b) unstable node; (c) hyperbolic point; (d) stable spiral point; (e) unstable spiral point; (f) elliptic point.

In the last three cases, whether the flow is clockwise or anticlockwise is determined from Eq. (11.53) by setting *dy* = 0 and *dx* > 0. If $\delta \dot{y} > 0$, the motion is upwards and hence anticlockwise; if $\delta \dot{y} < 0$, the motion is downwards, which corresponds to clockwise motion.

The cases we considered so far have nondegenerate roots l_1 and l_2. When the roots are degenerate, the general solution of Eq. (11.53) will be of the form $\delta \mathbf{X} = [c_1 \mathbf{D}_1 + c_2 (\mathbf{D}_2 + \mathbf{D}_1 \mathbf{t})] e^{\lambda t}$ (11.56) The sign and nature of the eigenvectors $\mathbf{D_1}$ and $\mathbf{D_2}$ decide the nature of the fixed points.

Case (vii): $\mathbf{D_2}$ is a null vector and $\mathbf{D_1}$ is arbitrary. The flow lines will be independent, intersecting straight lines forming a **stable star** if $l < 0$ and an **unstable star** if $l > 0$. These are represented in Figs. 11.7 (a) and 11.7 (b), respectively.

Case (viii): $\mathbf{D_2}$ is not a null vector; the flow lines will be curved, forming a **stable improper node** if $l < 0$, as shown in Fig. 11.7 (c) and an **unstable**

improper node if $l > 0$, as shown in Fig. 11.7 (d).

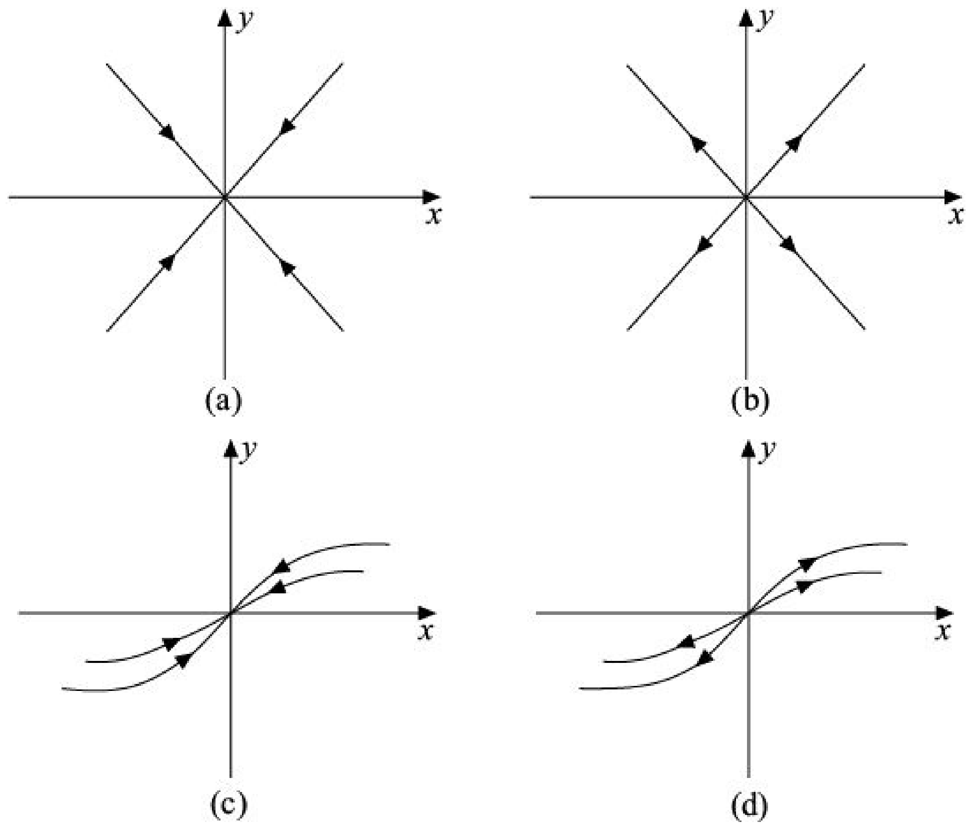


Fig. 11.7 Local phase flows for (a) stable star; (b) unstable star; (c) stable improper node; (d) unstable improper node.

11.9 FIXED POINT ANALYSIS OF A DAMPED OSCILLATOR

To illustrate the fixed point method of analysis, we discuss the case of a damped oscillator. The equation of motion of the damped harmonic oscillator is

$$\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0 \tag{11.57}$$

where $2b$ is the damping coefficient. Eq. (11.57) can be written as a pair of coupled first order equations:

$$\dot{x} = y \tag{11.58}$$

$$\dot{y} = -\omega_0^2 x - 2by \tag{11.59}$$

At the fixed point, $\dot{x} = \dot{y} = 0$. With this condition, from Eqs. (11.58) and (11.59), we have $x = 0$, $y = 0$. Hence, the only fixed point is $x_0 = y_0 = 0$. Writing the coupled equations denoted by Eqs. (11.58) and (11.59) in matrix form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{11.60}$$

The eigenvalues of the matrix are the roots of

$$\begin{vmatrix} -\lambda & 1 \\ -\omega_0^2 & -2b-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad \lambda^2 + 2b\lambda + \omega_0^2 = 0 \tag{11.61}$$

Solving

$$\lambda_1 = -b + \sqrt{b^2 - \omega_0^2} \qquad \lambda_2 = -b - \sqrt{b^2 - \omega_0^2} \tag{11.62}$$

Depending on the values of λ_1, λ_2 and ω_0, three possibilities arise:

Case (i): $b^2 > \omega_0^2$–Under this condition $\lambda_1 < \lambda_2 < 0$. Hence, the fixed point (0, 0) is a stable node.

Case (ii): $b^2 < \omega_0^2$; $\lambda_1 = \lambda_2^* = -b + i\sqrt{\omega_0^2 - b^2}$. Hence, the fixed point (0, 0) is a stable spiral.

Case (iii): $b^2 = \omega_0^2$; $\lambda_1 = \lambda_2 = -b$, which is a degenerate case. The (0, 0) fixed point then corresponds to a stable improper node.

11.10 LIMIT CYCLES

Apart from the simple fixed points (equilibrium points) in the linear stability analysis, a dynamical system may exhibit other type of stable solutions. These are the **limit cycles** that are characterized by periodically oscillating closed trajectories. A system exhibiting such a feature is the Van der Pohl oscillator

$$\ddot{x} - b(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{11.63}$$

which may be transformed into the following two-coupled differential equations of first order:

$$\dot{x} = y \tag{11.64}$$

$$\dot{y} = b(1 - x^2)y - \omega_0^2 x \tag{11.65}$$

This has a fixed point at (x, y) = (0, 0), which will be an unstable node if

$b^2 > 4\omega_0^2$ and an unstable spiral point if $b^2 < 4\omega_0^2$. (Problem 11.2). Considering the unstable spiral point, as x and y increase, the nonlinear term $-bx^2y$ dominates in Eq. (11.65), which suggests a decay back to the origin. That is, trajectories far from the origin move inwards. By continuity there must be at least one solution that stays in the middle. This solution is the **limit cycle** of the system (See Fig. 11.8), which is a closed orbit encircling the origin. Solutions starting either within or outside it are attracted to it but can never cross it. The exact shape of the limit cycle has to be worked out numerically.

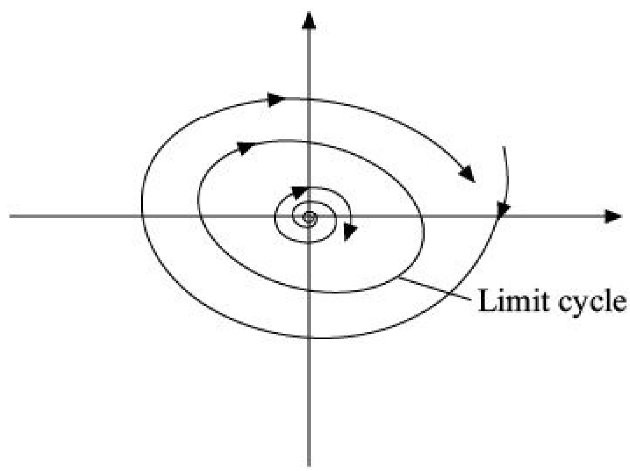


Fig. 11.8 The limit cycle.

In a **stable limit cycle** or **attracting limit cycle** all the trajectories approach the limit cycle, whereas in an **unstable limit cycle** or **repelling limit cycle** all the trajectories move away from it. If one set of trajectories (inside or outside) approaches the limit cycle and the other set moves away from it, the limit cycle is said to be a **semi-stable** or **saddle cycle**.

WORKED EXAMPLES

Example 11.1 Find the possible fixed points of a damped pendulum with damping force proportional to velocity. Discuss their stability.

Solution: The equation of motion of a damped pendulum is

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0 \qquad \text{(i)}$$

where $2b$ is the damping coefficient. Writing x for θ and y for $\dot{\theta}$, Eq. (i) can be written as

$$\dot{x} = y \qquad \text{(ii)}$$

$$\dot{y} = -\frac{g}{l}\sin x - 2by \qquad \text{(iii)}$$

At the fixed points, $\dot{x} = 0,\ \dot{y} = 0$. This gives $y = 0$ and $\sin x = 0$ or $x = \pm n\pi$. That is, the fixed points are $(x_n, y_n) = (\pm n\pi, 0)$, $n = 0, 1, 2, \ldots\ldots$ From Eqs. (ii) and (iii), we have

$$\delta\dot{x} = \delta y$$

$$\delta\dot{y} = -\frac{g}{l}\cos x\,\delta x - 2b\,\delta y$$

These equations can be written in the matrix form as

$$\frac{d}{dt}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{l}\cos x_n & -2b\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} \qquad \text{(iv)}$$

The eigenvalues λ_1 and λ_2 of the matrix are the solutions of

$$\begin{vmatrix}-\lambda & 1\\ -\frac{g}{l}\cos x_n & -2b-\lambda\end{vmatrix} = 0 \qquad \text{or} \qquad \lambda^2 + 2b\lambda + \frac{g}{l}\cos x_n = 0$$

$$\lambda_1 = -b + \sqrt{b^2 - \frac{g}{l}\cos x_n} \qquad \lambda_2 = -b - \sqrt{b^2 - \frac{g}{l}\cos x_n} \qquad \text{(v)}$$

At the fixed points, $(x_n,\, y_n) = [\pm(2n+1)\pi,\, 0]$

$$\lambda_1 = -b + \sqrt{b^2 + \frac{g}{l}} \qquad \text{and} \qquad \lambda_2 = -b - \sqrt{b^2 + \frac{g}{l}}$$

That is, $\lambda_2 < 0 < \lambda_1$, which corresponds to a hyperbolic point. At the fixed points $(x_n,\, y_n) = (\pm\, 2n\,\pi,\, 0)$

$$\lambda_1 = -b + \sqrt{b^2 - \frac{g}{l}} \qquad \text{and} \qquad \lambda_2 = -b - \sqrt{b^2 - \frac{g}{l}}$$

This leads to three different cases:

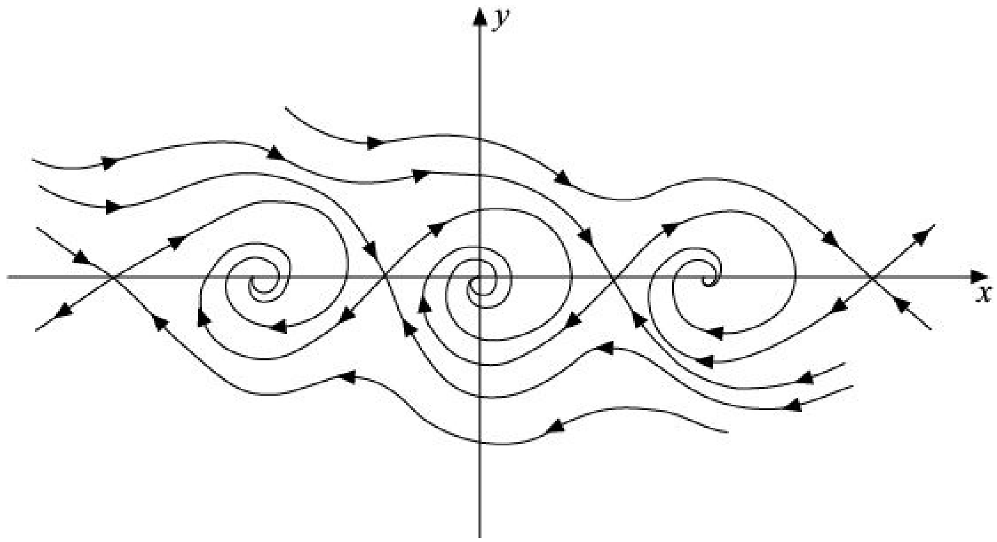


Fig. 11.9 Phase plane for damped pendulum for $b^2 < g/l$.

Case (i): $\boldsymbol{b^2 < g/l}$: $\lambda_1 = \lambda_2^* = -b + i\sqrt{\frac{g}{l} - b^2}$, which corresponds to a stable spiral point. To find the flow direction, on setting dy = 0 and dx > 0, Eq. (iv) gives $\delta \dot{y} < 0$. Hence, the flow direction is clockwise. The phase curve for this case is illustrated in Fig. 11.9.

Case (ii): $\boldsymbol{b^2 > g/l}$: This condition leads to $l_1 < l_2 < 0$, which corresponds to a stable node.

Case (iii): $\boldsymbol{b^2 = g/l}$: We have the degenerate case with $l_1 = l_2 = -b$ corresponding to a stable improper node.

Example 11.2 Consider the conservative nonlinear system of the coupled equations of Voltera in connection with the growth or decay of population of two species, one of which thrives on the other: $\dot{x} = x - xy$ and $\dot{y} = -y + xy$ Discuss the possible fixed points, their stability and the nature of phase curve.

Solution: At the fixed points, $\dot{x} = \dot{y} = 0$. Then $x = xy$ and $y = xy$. Hence, $x = y$. Substituting this condition in $x = xy$, we have $x = x^2$ which means $x = 0$ or 1. Then it is obvious that the system has two fixed points $(x_1, y_1) = (0, 0)$ and (x_2, y_2)
= (1, 1). From the given equations, we have $\delta \dot{x} = (1 - y)\,\delta x - x\delta y$

$$\delta \dot{y} = y\delta x + (-1 + x)\delta y$$

In the matrix form,

$$\frac{d}{dt}\begin{bmatrix}\delta x \\ \delta y\end{bmatrix} = \begin{bmatrix}1 - y_i & -x_i \\ y_i & -1 + x_i\end{bmatrix}\begin{bmatrix}\delta x \\ \delta y\end{bmatrix} i = 1, 2 \qquad \text{(i)}$$

For the fixed point (0, 0), the stability matrix reduces to $\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$ which has eigenvalues $\lambda = \pm 1$. Hence, it is a hyperbolic or saddle point. The eigenvectors of the matrix corresponding to +1 and –1 eigenvalues are $\begin{pmatrix}1 \\ 0\end{pmatrix}$ and $\begin{pmatrix}0 \\ 1\end{pmatrix}$, respectively. The general solution is then

$$\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = c_1\begin{pmatrix}1 \\ 0\end{pmatrix}e^{t} + c_2\begin{pmatrix}0 \\ 1\end{pmatrix}e^{-t} \qquad \text{(ii)}$$

where c_1 and c_2 are constants.

For the fixed point $(x_2, y_2) = (1, 1)$ the stability matrix reduces to $\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$ having eigenvalues $\lambda = \pm i$. Hence, the point (1, 1) is an elliptic point. The flow direction can be obtained by setting $dy = 0$ and $dx > 0$ in Eq. (i). When this condition is applied, we have $\delta \dot{y} > 0$. Hence, the rotation about the elliptic fixed point is anticlockwise. The approximate phase portrait is illustrated in Fig. 11.10.

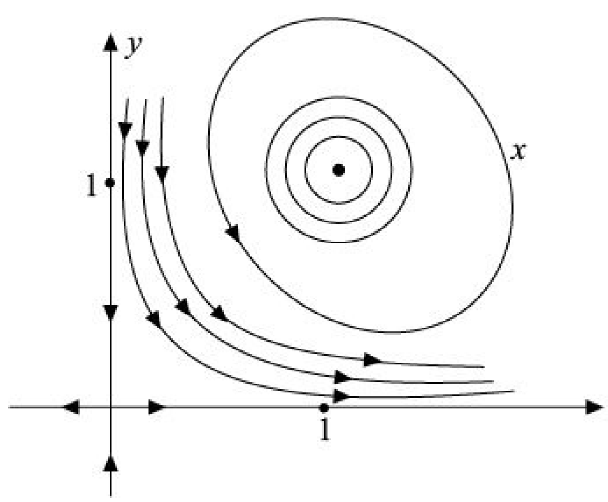


Fig. 11.10 Phase portrait of the coupled equations of Voltera **Example 11.3** Solve the nonlinear equation of motion $\ddot{x} + bx^3 = 0 \qquad b = \text{constant}$

which corresponds to the motion of a particle under a nonlinear restoring force. Hence, show that the period is explicitly dependent on the energy, given that $\int_0^a \frac{dx}{\sqrt{a^4 - x^4}} = \frac{\pi}{2a}$.

Solution: Given equation is $\ddot{x} + bx^3 = 0$, where *b* is a constant. (i) This equation can be written in the form of a pair of coupled first order equations

$$\dot{x} = y \quad \text{(ii)}$$

$$\dot{y} = -bx^3 \quad \text{(iii)}$$

Dividing Eq. (ii) by (iii) and rearranging

$$y\,\dot{y} + bx^3\,\dot{x} = 0$$

$$\frac{d}{dt}\left(\frac{y^2}{2} + \frac{bx^4}{4}\right) = 0 \quad \text{(iv)}$$

which is the first integral of Eq. (i). Since $y = \dot{x}$, the constant can now be identified as the total energy *E*. Then

$$\frac{\dot{x}^2}{2} + \frac{bx^4}{4} = E \qquad \text{or} \qquad \dot{x} = \sqrt{2\left(E - \frac{bx^4}{4}\right)}$$

$$dt = \frac{dx}{\sqrt{2\left(E - \frac{bx^4}{4}\right)}} = \frac{dx}{\sqrt{2E - bx^4/2}} \quad \text{(v)}$$

The classical turning point can be obtained by setting the potential energy to E when $\dot{x} = 0$. That is, $\frac{bx^4}{4} = E$ or $x = \pm\left(\frac{4E}{b}\right)^{1/4}$.

Integrating Eq. (v) and introducing a factor of 2 to take care of a full cycle, the period

$$T = 2 \int_{-(4E/b)^{1/4}}^{(4E/b)^{1/4}} \frac{dx}{\sqrt{2E - bx^4/2}}$$

Writing $\qquad a^2 = \sqrt{4E/b}$

$$T = \frac{2\sqrt{2}}{\sqrt{b}} \int_{-a}^{a} \frac{dx}{\sqrt{a^4 - x^4}} = \frac{4\sqrt{2}}{\sqrt{b}} \int_{0}^{a} \frac{dx}{\sqrt{a^4 - x^4}}$$

Substituting the value of the integral

$$T = \frac{4\sqrt{2}}{\sqrt{b}} \frac{\pi}{2a} = \frac{2\sqrt{2}}{a\sqrt{b}}\pi$$

Replacing a

$$T = \frac{2\pi}{(bE)^{1/4}}$$

That is, the period is explicitly dependent on energy.

REVIEW QUESTIONS

1. Explain integration of linear second order equations by the method of quadrature with the help of an example.
2. What is a phase curve? Illustrate diagrammatically the phase curve of a simple harmonic oscillator.
3. Explain the phase portrait of a damped oscillator.
4. With the necessary diagram, explain the phase portrait of a pendulum.
5. The existence of a constant first integral provides a definite geometrical constraint on the phase flow. Substantiate.
6. Draw the phase curve of a simple pendulum and match it with the curve

representing the potential.

7. Explain how an approximate phase portrait is built up for nonconservative systems.
8. Explain the following with the local phase flow curves: (i) stable node (ii) unstable node (iii) hyperbolic point (iv) stable spiral point (v) unstable spiral point (vi) elliptic point.
9. What are (i) stable stars (ii) unstable stars (iii) stable improper nodes (iv) unstable improper nodes?
10. Outline the fixed point stability analysis of the damped linear harmonic oscillator.
11. What are limit cycles? Distinguish between stable limit cycle and semi-stable limit cycle.

PROBLEMS

1. Discuss the fixed point analysis of a pendulum and show that the fixed points are either elliptical or hyperbolic.

2. Investigate the Van der Pohl oscillator equation

$$\ddot{x} - b(1 - x^2)\dot{x} + \omega_0^2 x = 0$$

and show that $(x, y) = (0, 0)$ is a fixed point which will be an unstable node if $b^2 > 4\omega_0^2$ and an unstable spiral point if $b^2 < 4\omega_0^2$.

3. A one-dimensional system is described by the following equation of motion:

$$\ddot{x} + \alpha\dot{x} + \beta x + \gamma x^3 = 0$$

Discuss the possible fixed points and their stability.

12

Classical Chaos

At times, sudden and exciting changes in nonlinear systems may give rise to the complex behaviour called **chaos**. Chaotic systems, although deterministic, exhibit extensive randomness. Chaotic trajectories arise from the motion of nonlinear systems, which are nonperiodic but still somewhat predictable. Such complicated chaotic dynamics can be present even in deceptive simple systems. The existence of chaotic dynamics in mathematics dates back to the days of Poincare at the turn of the 20th century. However, only in the second half of the 20th century has the wide-ranging impact of chaos in the different branches of physics been visualized. We shall discuss here some of the definitions, concepts and results that are basic to the field of chaotic dynamics.

12.1 INTRODUCTION

Chaos describes the complicated behaviour of dynamical systems that lies between regular deterministic trajectories and a state of noise. It is used to describe the time behaviour of a system when the behaviour is aperiodic, that is, when it never repeats exactly. Chaos shows up in systems that are essentially free from noise and are relatively simple. The three major components that determine the behaviour of a system are the time evolution equations, the values of the parameters describing the system and the initial conditions. For example, a small change in one or more of the initial conditions could completely alter the course of the system, and the final outcome cannot be predicted.

The theoretical framework needed to describe chaos and chaotic systems is beyond the scope of this book. However, certain qualitative features will be developed without introducing serious mathematics. The description of the behaviour of the system will be described under **phase space** or **state space.**

12.2 BIFURCATION

Bifurcation means splitting into two parts. The behaviour of dynamical systems is influenced by the value of one or more control parameters (*m*). The control parameters could be the amount of friction, the strength of an interaction, the amplitude and frequency of a periodic perturbation or some other quantity. The term *bifurcation* is generally used in the study of nonlinear dynamics to describe the change in the behaviour of the system as one or more control parameters are varied. The control parameter may suddenly change a stable equilibrium position into two such positions, or a system initially at rest may begin to oscillate. The phenomenon of additionally arising solutions or of solutions that suddenly change their character is called **branching** or **bifurcation**. Bifurcation for the logistic map function is discussed in Sections 12.3 and 12.5, and the diagram is represented in Fig. 12.6.

If the behaviour of a system in the neighbourhood of an equilibrium solution is changed, it is called a **local bifurcation**. If the structure of the solutions is modified on a larger scale, it is called **global bifurcation**. In the simple systems we study here, we encounter only the bifurcations generated by a single control parameter.

12.3 LOGISTIC MAP

To understand the phenomenon of chaos, we consider a simple mathematical model used to describe the growth of biological populations. If N_0 is the number of insects born at a particular time and N_1 is the number that survives after one year, the simplest way one can relate the two is to write $N_1 = AN_0$ (12.1)

where *A* is a number that depends on the conditions such as food supply, weather conditions, water, *etc*. Suppose that *A* remains the same for the subsequent generation. If $A > 1$, the number will increase year after year, leading to an explosion. If $A < 1$, the number will decrease and end up with extinction. To limit the growth of population, we have to incorporate another term that would be insignificant for small values of *N* but becomes significant as *N* increases. One possible way is to add a term of the type BN_0^2, where *B* is very small. The effect of this term should be to decrease the population. In such a case Eq. (12.1) becomes

$$N_1 = AN_0 - BN_0^2 \qquad B << A \tag{12.2}$$

The number in the subsequent years then changes as follows:

$$N_2 = AN_1 - BN_1^2 \tag{12.3a}$$

$$N_3 = AN_2 - BN_2^2 \tag{12.3b}$$

In order to have $N_{n+1} > 0$, N_n cannot exceed a value, say N_{max}. That is,

$$AN_{max} - BN_{max}^2 \geq 0 \qquad \text{or} \qquad N_{max} \leq \frac{A}{B} \tag{12.4}$$

Let us introduce a new variable x_n defined by

$$x_n = \frac{N_n}{N_{max}} \tag{12.5}$$

It means that x_n is the population in the nth year as a fraction of N_{max}. It is obvious that the value of x is in the range 0 1. With this definition, Eq. (12.2)

$$x_{n+1}N_{max} = Ax_nN_{max} - B(x_n N_{max})^2$$

gives

$$= Ax_nN_{max} - \frac{A}{N_{max}}(x_nN_{max})^2$$

$$x_{n+1} = Ax_n(1 - x_n) = f_A(x) \tag{12.6}$$

where Eq. (12.4) is used. Eq. (12.6) is an important one as it played a crucial role in the development of chaos. The function $f_A(x)$ is called the **iteration function** since the population fraction in subsequent years can easily be obtained by iterating the mathematical operations as per Eq. (12.6). The plot of the function $f_A(x)$ versus x for four values of the parameter A is illustrated in Fig. 12.1. In the figure the diagonal line is the plot of $f_A(x) = x$.

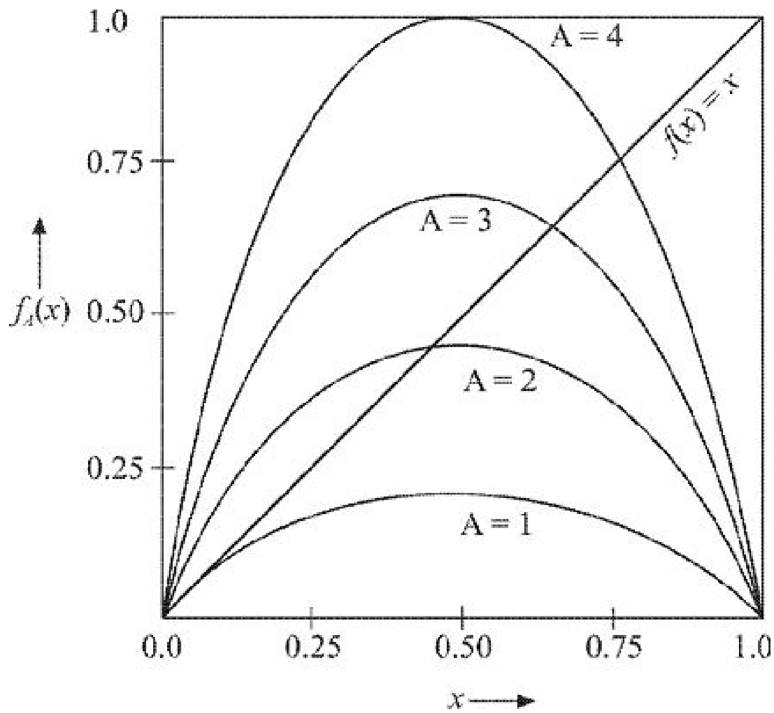


Fig. 12.1 Function $f_A(x)$ versus x for various values of A.

Since the environment represented by the parameter A remains a constant over a period of time, we expect the value of population fraction x also to settle down into some definite value. This value of x may change gradually if A changes gradually. Hence, if we start with some value of x_0.

$x_1 = f_A(x_0)$ $x_2 = f_A(x_1)$ $x_3 = f_A(x_2)$, (12.7) The function $f_A(x)$ is therefore referred to as the **iterated map function** or the **logistic map function**.

The nonlinear equation, Eq. (12.6), is a parabola since it is in the form $y = bx - cx^2$. The maximum value exceeds unity if $A > 4$. If $A \leq 1$, $x_1, x_2, \ldots$, tend to zero, which means the population is heading towards extinction. Hence, we need consider only the range of values $1 \leq A \leq 4$.

The successive terms $x_1, x_2, x_3, \ldots$ generated by this iteration procedure is called the **trajectory** or **orbit**. The first few points of a trajectory depend on the starting value of x. The subsequent behaviour is the same for almost all starting points for a given value of A. An x-value, called x^*, defined by the condition

$$x_A^* = f_A(x_A^*)$$

is called a **fixed point** of the iterated map. The subscript A to x^* is to indicate that it depends on A. The fixed point x_A^* means that successive iterations are expected to bring x_{n+1} closer and closer to the limiting value x_A^* and further iterations produce no additional change in x_n. For the logistic map, the fixed points can easily be determined by solving the equation

$$x^* = Ax^*(1-x^*) \qquad \text{or} \qquad x^*(Ax^* - A + 1) = 0$$

$$x_A^* = 0 \qquad \text{and} \qquad x_A^* = 1 - \frac{1}{A} \tag{12.8}$$

That is, in general there are two fixed points in the range $0 \le x \le 1$. For $A < 1, x_A^* = 0$ is the only fixed point in this range. However, for $A > 1$, both the fixed points fall in this range. In Fig. 12.1, whenever the $f_A(x)$ curve crosses the diagonal line, the map function has a fixed point.

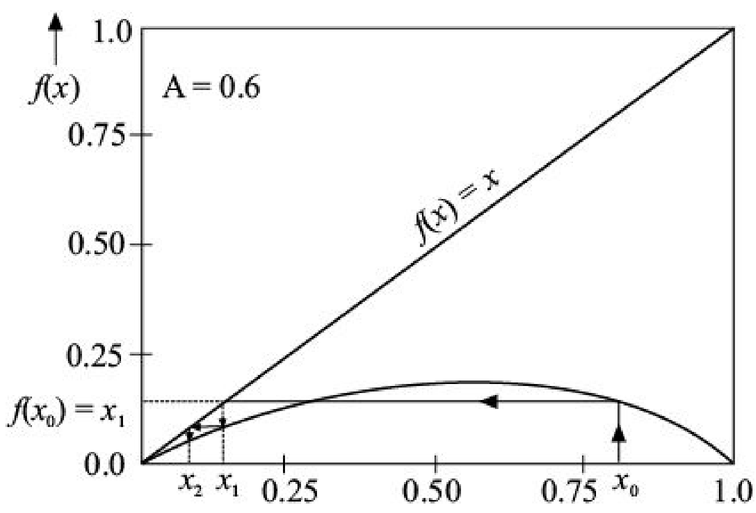


Fig. 12.2 Representation of the iteration in Eq. (12.6) starting from $x_0 = 0.8$ and $A = 0.6$.

Next we shall see how a trajectory that starts from some value of x different from zero approaches zero if $A < 1$. Fig.12.2 gives the graphic representation of the iteration map starting from $x_0 = 0.8$ with $A = 0.6$. From the starting value of $x_0 = 0.8$ on the x-axis, draw a vertical which meets the f_A curve. The intersection value determines the value of x_1. From the intersection point, draw a line parallel to the x-axis to the diagonal. Directly below this intersection point is x_1 on the

x-axis. From the intersection point on the diagonal, draw a vertical to the f_A

curve. The resulting intersection point on the f_A curve determines x_2 (see Fig. 12.2). Continuing this process we approach the final value x = 0. In general, we conclude that if $A < 1$ the population dies out, that is, x 0 as n increases.

Next, we shall consider the situation when $A > 1$. Fig.12.3 illustrates the plot of f_A (x) against x along with the diagonal line and the trajectory starting at

x = 0.08. From the figure we see that the trajectory is heading for the fixed point $x_A^* = 1 - (1/A) \cong 0.47$. It may be noted that all trajectories starting in the range $0 < x < 1$ approach this same attractor (refer to Section 12.4). It may also be noted that for $A > 1$, $x^* = 0$ has become a repelling fixed point since trajectories starting near $x = 0$ also move away from the value. In Figs. 12.4 and 12.5 we have plotted f_A (x) versus x for A slightly less than 3 (A = 2.9) and for slightly greater than

3, respectively. In the case given in Fig. 12.4 ($A < 3$) a spiralling of the trajectories on to a stable fixed point x^* is noticed, whereas in Fig.12.5 ($A > 3$) a spiralling away of the trajectories from an unstable fixed point x^* takes place.

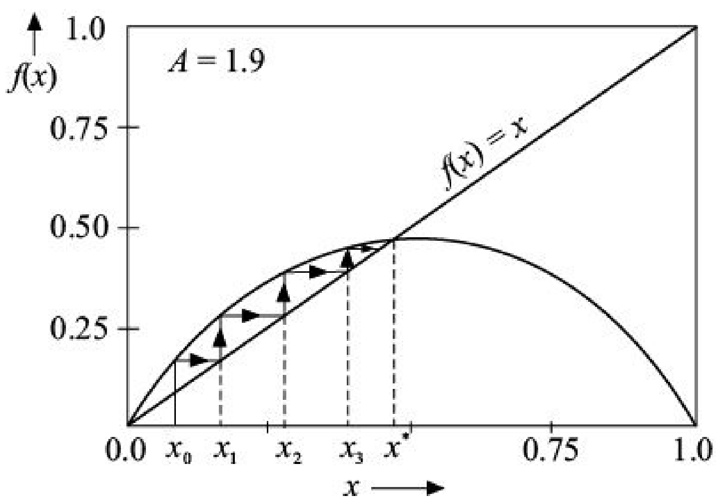


Fig. 12.3 Graphic representation of the iteration procedure with A = 1.9 and x_0 = 0.08 leads to

$x^* \cong 0.47$.

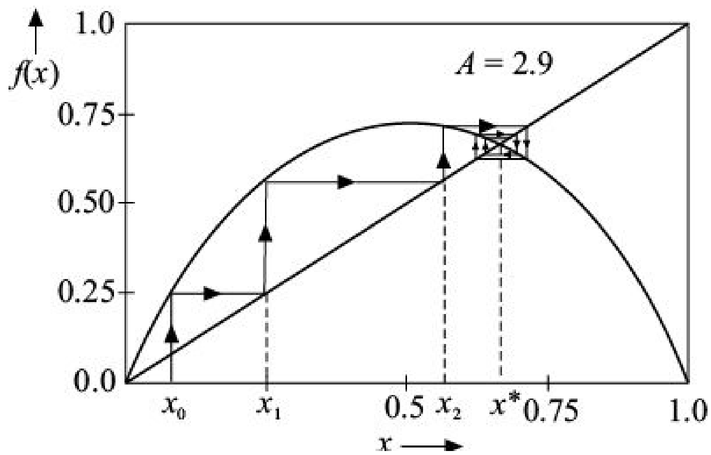


Fig. 12.4 Graphic representation of the iteration procedure with A = 2.9 and x_0 = 0.08. Note the spiralling of the trajectories on to the stable fixed point $x^* \cong 0.66$.

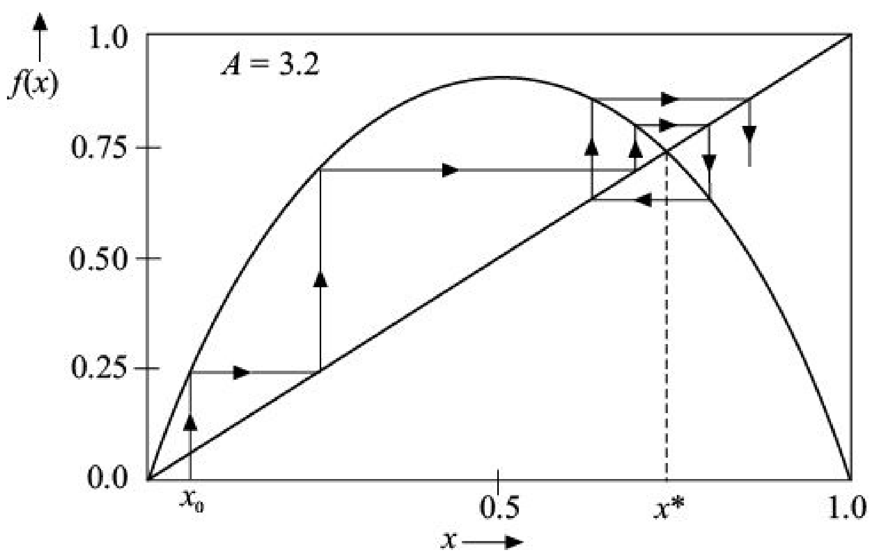


Fig. 12.5 Graphic representation of the iteration procedure with A = 3.2. Note the spiralling away of the trajectories from the unstable fixed point x^*.

From what we have seen, for a stable fixed point A must satisfy the condition $1 < A < 3$. Such a fixed point constitutes an attractor since values of x_n iterate toward it. When A is slightly greater than 3, it is noticed that the trajectory does not settle down to a single attracting value but alternates between the two values or attractors.

$x_n = 0.559$ and $x_{n+1} = 0.764$ (12.9) This is often referred to as *period-2 behaviour*. In other words, at $A = 3$ a **period doubling bifurcation** occurs. The point at which the bifurcation takes place is called a **critical point**. For the parameter $A = 3.5$, a double bifurcation corresponding to a fourfold cycle involving the four attractors occurs.

$x_n = 0.501$ $x_{n+1} = 0.875$ $x_{n+2} = 0.383$ $x_{n+3} = 0.827$

As A is increased there is an eightfold cycle for $A = 3.55$, a sixteenfold cycle for $A = 3.566$, ... This continues until the value.

$A = 3.5699456$ (12.10) called the **Feigenbaum point** is reached. Beyond A, the behaviour becomes chaotic. Fig. 12.6 is a plot of x against A which illustrates the bifurcations. These are sometimes referred to as **Feigenbaum diagrams**. For values of parameter A beyond A, successive x_n terms generate all possible random values. In this chaotic region, two points that are initially very close generate successive sequences that do not remain near each other. As a consequence, the region in the (A, x) plane beyond A is extremely densely populated. That is, we have an attractor of an infinite set of points. Another interesting property of the diagram is the presence of nonchaotic windows embedded in the chaos. Odd cycles (e.g.,3 cycles) also appear in the chaotic regime.

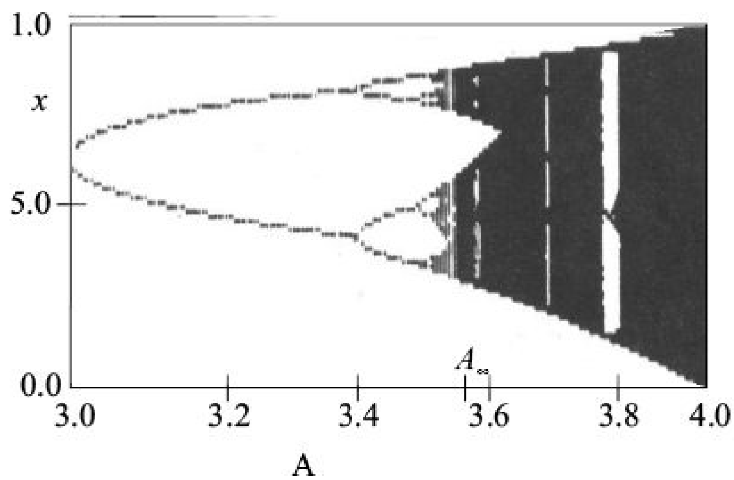


Fig. 12.6 Bifurcation diagram for logistic map function.

12.4 ATTRACTORS

In the section on logistic maps (Section 12.3), we have seen that (i) if $A < 1$, all trajectories starting in the range $0 < x < 1$ approach the fixed point zero, (ii) if

$A > 1$, trajectories starting in the range $0 < x < 1$ approach the same fixed point $x_A^* = 1 - (1/A)$. Such a fixed point constitutes an attractor since the values of x_n iterate toward it. In general, the **attractor** is that set of points in phase space to which the solution of an equation evolves long after transients have died out. As an example, consider the state of the damped harmonic oscillator

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad (12.11)$$

The phase space variables specifying the state of the oscillator are the position x and velocity v = (dx/dt). Any initial condition eventually comes to rest at the point (x, v) = (0, 0), which is the attractor for the system. Here the attractor is a single point.

In a dynamical system, if trajectories originating from starting values of the parameter x between 0 and 1 approach the final value, say $x = 0.47$, then that point is called the attractor for those orbits. The interval $0 \leq x \leq 1$ is called the **basin of attraction** for that attractor since any trajectory starting in that range approaches $x = 0.47$. In Section 11.10, we considered a system in which the initial conditions start the motion on a trajectory that does not lie on a stable path but that evolves towards a stable orbit in phase space called a *limit cycle*. The limit cycle is an example of an attractor. For chaotic systems, the attractor can be geometrically much more complicated.

If the attractor is a fixed point, we say its dimensionality d_A is 0 since a point is a 0-dimensional object in geometry. If the attractor is a line or a simple closed curve, its dimensionality $d_A = 1$, since a line or a curve is a 1-dimensional object. Extending this nomenclature, a surface has a dimensionality $d_A = 2$, a solid volume a dimensionality $d_A = 3$. We can have hypervolumes of higher dimensions also. The dimensionality of an attractor gives us an idea of the number of active degrees of freedom for the system.

In nonlinear dynamics there is a different type of attractor, called **strange attractor,** whose dimensionality is not an integer. A familiar example is the one associated with the logistic map (Section 12.3) in the region where the parameter $A > 3.5699$. In that region the map becomes the chaotic occurrence of a strange attractor in a dynamical system, which is an indication that the system is chaotic. Hence, strange attractors are also called **chaotic attractors**.

12.5 UNIVERSALITY OF CHAOS

In the section on logistic maps (Section 12.3) we discussed the basics of the period doubling route to chaos. From a study of period doubling, Feigenbaum discovered that there might be some universality underlying the phenomenon of chaos. He studied the bifurcation diagram for the logistic maps of the two functions

$$x_{n+1} = Ax_n(1 - x_n) \tag{12.12}$$

and,

$$x_{n+1} = B\sin(\pi x_n) \tag{12.13}$$

and found the same rate of convergence for both the maps.

To understand more about convergence and other details, let us consider the logistic map equation, Eq. (12.12), which is the same as Eq. (12.6). A portion of the bifurcation diagram of the logistic map in Fig. 12.6 is reproduced in Fig. 12.7. In the figure, A_1 is the parameter value where period-1 gives birth to period-2, A_2 is the value when period-2 changes to period-4, and so on. Denoting the parameter value at which period-2^n is born by A_n, let us examine the ratio

$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n} \tag{12.14}$$

Feigenbaum found that this ratio is approximately the same for all values of *n*. Surprisingly, for large values of *n* this ratio approached a number, called **Feigenbaum** *d*, that was the same for both the map functions defined by Eqs. (12.12) and (12.13).

$$\delta = \lim_{n\to\infty} \delta_n = 4.66920161 \tag{12.15}$$

Later, he was able to establish the same convergence ratio for the iterated map function that has a parabolic shape near its maximum value.

As part of the numerical investigation of mapping functions, Feigenbaum introduced the **Feigenbaum** *a* by the relation

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 2.5029 \tag{12.16}$$

where d_n is the size of the bifurcation pattern of period 2^n just before it gives birth to period 2^{n+1} (see Fig.12.7). The ratio involves the *d*'s for the corresponding parts of the bifurcation pattern. The theory leading to the number 2.5029 applies only in the limit of higher order bifurcation. The agreement with experiment is therefore only to be expected.

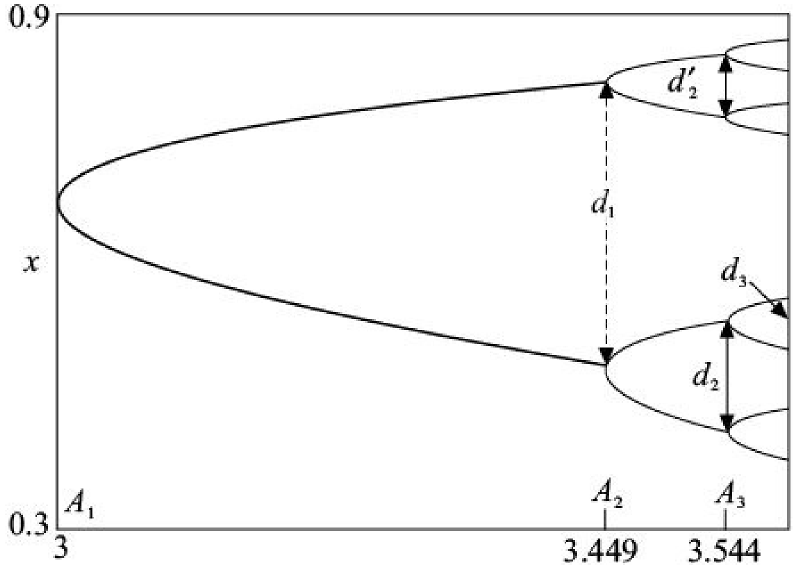


Fig. 12.7 A portion of the bifurcation diagram for the logistic map function given in Eq. (12.12).

Subsequently, universality has been discovered in other nonlinear systems also. It may be mentioned here that these features do not hold good for all nonlinear systems.

12.6 LYAPUNOV EXPONENT AND CHAOS

So far we have been discussing chaos in a qualitative way. In this section, a method of quantifying the chaotic behaviour is introduced. Consider a one-dimensional state space and let x_0 and x be two nearby points. Let $x_0(t)$ be a trajectory that arises from the point x_0 while $x(t)$ be one that arises from x. The distance between the two trajectories $s = x(t) - x_0(t)$ (12.17) grows or contracts exponentially in time. The time rate of change of distance between the trajectories is

$$\dot{s} = \dot{x} - \dot{x}_0 \tag{12.18}$$

The time development equation be

$$\dot{x}(t) = f(x) \qquad \dot{x}_0(t) = f(x_0) \tag{12.19}$$

By a Taylor series expansion

$$f(x) = f(x_0) + \left.\frac{df(x)}{dx}\right|_0 (x - x_0) + \text{higher order terms}$$

Neglecting higher order terms

$$\dot{x}(t) = \dot{x}_0(t) + \left.\frac{df(x)}{dx}\right|_0 (x - x_0) \tag{12.20}$$

$$\dot{s} = \left.\frac{df(x)}{dx}\right|_0 (x - x_0) = \left.\frac{df(x)}{dx}\right|_0 s \tag{12.21}$$

Writing

$$\lambda = \left.\frac{df}{dx}\right|_{x=0} \tag{12.22}$$

$$\frac{ds}{s} = \lambda dt \qquad \text{or} \qquad \ln s = \lambda t + \text{constant } C$$

$$s = C\, e^{\lambda t} \tag{12.23}$$

Since the constant $C = s(t = 0)$

$$s(t) = s(t = 0)\, e^{\lambda t} \tag{12.24}$$

That is, the distance between the two changes exponentially with time. If *l* is positive the two trajectories diverge, and if *l* is negative the two trajectories converge. For *l* > 0, the motion will be chaotic, and **Lyapunov exponent** *l* quantifies the average growth of an infinitesimally small deviation of a regular orbit arising from perturbation. For *t* >> (1/*l*), chaos is appreciable.

In species with two or three dimensions, we may define Lyapunov exponent for each of the directions. If the system evolves by an iterative process instead of a temporal process, then Eq. (12.24) assumes the form

$$s(n) = s_0 e^{n\lambda} \tag{12.25}$$

where *n* is the number of iterations and the exponent *l* is now dimensionless.

12.7 FRACTALS

In Fig. 12.7 we note that the lower section between A_2 and A_3 looks like the region between A_1 and A_2 when the parameter axis between A_2 and A_3 is expanded by a factor d and the vertical axis for that region is expanded by a factor a. The upper portion requires a different degree of magnification. A geometrical structure having this replicating behaviour under magnification is said to be **self-similar**. Such self-similar objects are referred to as **fractals** since their geometric dimension is often a fraction, not an integer.

The geometrical construction of a fractal is based on a simple iteration rule that is applied repeatedly. In the limit of infinite number of iteration steps, the fractal arises. We shall consider two of the familiar examples, the Cantor set and the Sierpinski gasket.

Cantor set: Consider a line segment, remove its middle third and get two line segments. Then remove the middle third in each of these two line segments to get a total of four. When this procedure is continued infinite times, there arises a series of dots with characteristic spacings, which is called the Cantor set. The formation of the Cantor set is illustrated in Fig. 12.8. At various stages in its generation, the set is self-similar since magnification of the set at the latter stages of generation have the same appearance as the set itself at the earlier stages of formation.

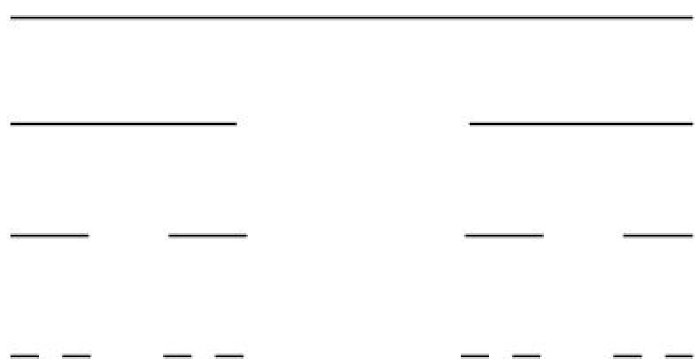


Fig. 12.8 The iterative construction of the Cantor set.

Sierpinski gasket: An equilateral triangular area forms the basic element of the Sierpinski gasket. The iteration rule is to subdivide each triangle into 4 congruent parts and remove the central one. Its self-similarity is obvious. Fig. 12.9 illustrates the iterative construction of the Sierpinski gasket.

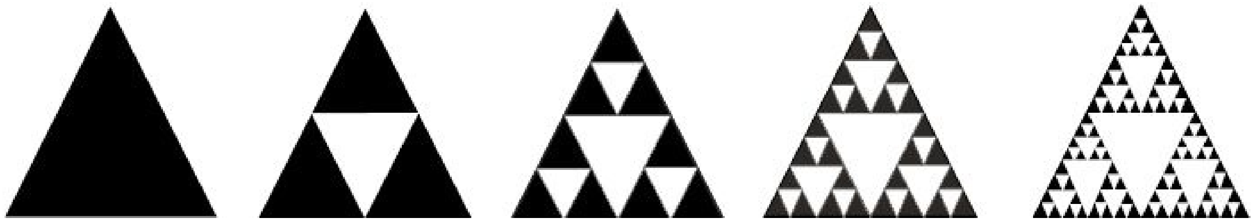


Fig. 12.9 Iterative construction of the Sierpinski gasket.

Fractal Dimension A number of methods are available for determining fractal dimension. However, no method is foolproof. Fractals and their dimensionalities play a crucial role in the dynamics of chaotic systems. For ordinary objects such as smooth curves, areas and volumes, the dimension is obvious as it coincides with visual conception. However, fractals behave differently. To start with, let us consider the dimensionality d_E in ordinary Cartesian or Euclidean space. In one dimension, consider a line segment of length a_0 divided into a large number of equal smaller parts, each of length $a \ll a_0$. In two dimensions, it is the subdivision of a square of side a_0 into many equal smaller parts, each of side a. In three dimensions, a cube of side a_0 is subdivided into number of equal cubes, each of side $a \ll a_0$. Denoting the total number of smaller parts by $N(a)$, we have

$$N(a) = \left({}^{a_0}\!\!\diagup_{a} \right)^{d_E} \qquad (12.26)$$

For the cases we considered above, it is obvious that $d_E = 1, 2$ or 3, respectively. From Eq. (12.26) we have

$$d_E = \frac{\ln N(a)}{\ln\left({}^{a_0}\!\!\diagup_{a} \right)}$$

$$= \frac{\ln N(\epsilon)}{\ln\left({}^{1}\!\!\diagup_{\epsilon} \right)} \qquad \epsilon = a/a_0 \qquad (12.27)$$

This formula for the dimension d_E is found to be applicable for fractal dimension d_F also. Sometimes, fractal dimension is also referred to as **capacity dimension** or **box dimension**. The fractal dimension d_F is then given by

$$d_F = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln\left({}^{1}\!\!\diagup_{\epsilon} \right)} = \lim_{n \to \infty} \frac{\ln N(\epsilon_n)}{\ln\left({}^{1}\!\!\diagup_{\epsilon_n} \right)} \qquad (12.28)$$

Here, in general, the number *N*(Œ) depends on the box size *Œ*.

As an example, we shall evaluate d_F for the Cantor set. For the Cantor set, after the first division, the length $a_1 = a_0/3 = a_0/3^1$, $a_2 = a_0/9 = a_0/3^2$, $a_3 = a_0/27 = a_0/3^3$, and so on. After the *n*th division $a_n = a_0/3^n$. After each subdivision, the number always doubles. Thus, $N(a_0) = 1$, $N(a_1) = 2 = 2^1$, $N(a_2) = 4 = 2^2$, $N(a_3) = 8 = 2^3$, and so on. Consequently,

$$d_F = \lim_{n \to \infty} \frac{\ln 2^n}{\ln 3^n} = \frac{\ln 2}{\ln 3} = 0.6309 \qquad (12.29)$$

For the Sierpinski gasket

$$d_F = \lim_{n \to \infty} \frac{\ln 3^n}{\ln 2^n} = \frac{\ln 3}{\ln 2} = 1.5849 \qquad (12.30)$$

All fractals are not similar. The self-similar objects form a simple class of fractals. In a different class of fractals, called **self-affine fractals,** their self-similarity is apparent only if different magnification factors are used in different directions. The fractal set generated by stochastic processes are referred to as

random fractals.

12.8 ROUTES TO CHAOS

Having understood the basics of chaos, we can now try to discuss the following routes or transitions to chaos: (i) Period doubling (ii) Quasi-periodicity (iii) Intermittency (iv) Crises.

More routes to chaos are undoubtedly there and more will undoubtedly be recognized when we learn systems with larger number of degrees of freedom. An experimental system may change its behaviour from regular to chaotic as the control parameters of the system are slowly changed. A given system may exhibit different types of routes to chaos for different ranges of parameter values.

Period Doubling As already discussed, an important route to chaos is that of period doubling bifurcations. This behaviour is exhibited by the one-dimensional nonlinear mappings of the form $x_{n+1} = f(x_n)$ (12.31) with $f(x_n)$ satisfying certain conditions. Of these mappings, an important one is the logistic map which we discussed in Section 12.3

$x_{n+1} = Ax_n(1 - x_n)$ $0 < x < 1$ (12.32) where A is an adjustable parameter. We have seen that for small values of A all iterates converge on to a single limit point. This behaviour continues till A reaches the value 3. At $A = 3$ a period-2 bifurcation occurs. As A is increased further, the period-2 bifurcates into a period-4 cycle ($A = 3.5$) which subsequently bifurcates into a period-8 cycle ($A = 3.55$), and so on. The values at which bifurcations occur become closer, converging to a critical value $A = 3.5699456$. At this point the orbit becomes aperiodic. Beyond this point the chaotic orbits, period-3 and other odd-period cycles start to appear, giving rise to the long time chaotic behaviour of the system. For A beyond the value at which a 3-cycle is born, each fixed point of the 3-cycle bifurcates into a pair of fixed points, one stable and the other unstable. Such bifurcations are called **saddle-node** bifurcations.

Driven nonlinear oscillators, Rayleigh-Benard instability of convective turbulence, chemical reactions, *etc.* are some of the physical examples of period doubling.

Quasi-periodicity A type of motion which is possible in three-dimensional state space is the quasi-periodic one. It is called quasi-periodic since it has two different frequencies associated with it. It can be analysed into two independent periodic motions with the trajectories constrained to the surface of a torus in the 3-dimensional state space. (A torus has the shape of a doughnut or the inner tube of a motor car wheel.) A mathematical description of quasi-periodic motion is given by

$$x_1 = (R + r\sin\omega_r t)\cos\omega_R t$$

$$x_2 = r\cos\omega_r t \qquad (12.33)$$

$$x_3 = (R + r\sin\omega_r t)\sin\omega_R t$$

where w_R and w_r are the two angular frequencies. Eq. (12.28) describes the motion on the surface of a torus whose larger radius is R and whose cross-sectional radius is r. The angular frequency w_R corresponds to the rotation around the large circumference, while the angular frequency w_r corresponds to rotation about the cross-section.

If the ratio of the two frequencies (w_r/w_R) can be expressed as a rational fraction such as *n/m*, then the type of motion is called **frequency-locked**, since an integral multiple of one frequency is equal to another integral multiple of the other. In this case the motion will eventually repeat itself and they are usually referred to as *closed orbits*. If the frequencies are not related rationally, the motion never exactly repeats itself. Such orbits are usually termed **quasi-periodic**. Thus, a single orbit will eventually cover the torus uniformly. Under certain circumstances, if the control parameter is changed further, the motion becomes chaotic. This route to chaos is sometimes referred to as the *Ruelle-Takens route* since its theoretical possibility was first suggested by them in 1971.

In the quasi-periodic process, as the control parameter is changed one might expect a long sequence of different frequencies as a mechanism for producing chaos in the system, as Landau had proposed such an infinite sequence of frequencies as the mechanism for producing fluid turbulence. However, at least in a number of cases it has been found that the system becomes chaotic instead of introducing a third distinct frequency for its motion.

The quasi-periodic process involves competition between two or more independent frequencies characterizing the dynamics of the system at least in two different ways. In one, a nonlinear system with a natural oscillation frequency is driven by an external periodic force and the competition is between these two frequencies. In the other, spontaneous oscillations develop at two or more frequencies as some of the parameters of the system is varied. In this case we have competition among the different modes of the system itself. In both cases, as these frequencies compete with each other the result may be chaos.

Some of the physical systems that display the quasi-periodic transition from regular to chaotic behaviour are the periodically perturbed cardiac cells, periodically driven relaxation oscillators, turbulence in a fluid flow confined between two coaxial cylinders with the inner cylinder rotating, *etc.*

Intermittency

Intermittency occurs whenever the behaviour of a system switches to and fro between two qualitatively different behaviours, even though all the control parameters remain constant and the external noise is absent. Though the system is described by deterministic equations, switching behaviour is random. Two types of intermittency are important. In the first type, the system's behaviour seems to switch between periodic and chaotic behaviours. For some control parameter value, let the behaviour of the system be predominently periodic with occasional bursts of chaotic behaviour. As the control parameter value is changed, the time spent being chaotic increases till finally the behaviour becomes chaotic all the time. As the parameter is changed in the opposite direction, the time spent in the periodic state increases and at some value the behaviour is completely periodic throughout. In the second type of intermittency, the system's behaviour seems to switch between periodic and quasi-periodic behaviours.

Crises

A **crisis** is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly disappear or suddenly change in size as some control parameter is adjusted. The type that disappears is called **boundary crisis**. The sudden expansion or contraction of a chaotic attractor is called an **interior crisis**. The appearance or sudden enlargement of fractal structure in a basin boundary is called **metamorphosis**.

The behaviour of a system at a crisis event can be illustrated with the help of Fig.12.6. In the figure, the bifurcation diagram suddenly ends at $A = 4$. The chaotic attractor which is present for A values just below 4 disappears in a boundary crisis.

The subject of chaos, as mentioned in the beginning, is introduced in a qualitative way. It is hoped that what is needed for understanding the basics of chaos has been conveyed.

REVIEW QUESTIONS

1. Explain bifurcation with the help of a diagram. What is period-doubling bifurcation?
2. What is a logistic map? What are the fixed points of an iterated map?
3. Explain with an example the period doubling route to chaos.
4. What is a Feigenbaum diagram ? What are Feigenbaum *d* and *a* ?
5. Explain the concept of attractors in chaos. What is a basin of attraction? What are strange attractors?
6. Write a note on fractals.
7. What is a Lyapunov exponent? How is it related to chaos?
8. Explain the quasi-periodicity route to chaos.
9. How are fractal dimensions determined?

Appendix A Elliptic Integrals

Elliptic integrals of the **first kind** are the integrals

$$F(x,k)=\int_0^x \frac{dx'}{\sqrt{(1-x'^2)(1-k^2x'^2)}} \qquad k^2<1 \tag{A.1}$$

An alternative form can easily be obtained from Eq. (A.1) by substituting

$$x'=\sin\theta' \qquad dx'=\cos\theta'\,d\theta'$$

$$F(\theta,k)=\int_0^\theta \frac{\cos\theta'\,d\theta'}{\cos\theta'\sqrt{1-k^2\sin^2\theta'}}=\int_0^\theta \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}} \tag{A.2}$$

Thus, Eqs. (A.1) and (A.2) are equivalent forms of elliptic integral of the first kind. The inverse of the elliptic integral in Eq. (A.2) or Eq. (A.1) are the Jacobi elliptic functions. To understand elliptic functions, write the elliptic integral in Eq. (A.2) as

$$u=\int_0^\theta \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}} \tag{A.3}$$

Evaluation of the integral in Eq. (A.3) for $k=0$ gives

$$u=\int_0^\theta d\theta'=\theta=\sin^{-1}x \tag{A.4}$$

The same result can also be obtained from Eq. (A.1), since when $k=0$ the integral is just the arc sin integral:

$$u=\sin^{-1}x=\sin^{-1}\sin\theta=\theta$$

$$\sin u=\sin\theta=x \tag{A.5}$$

Let us now consider the situation when *k* π 0. When *k* π 0, the integral is not a simple one. It will be a complicated function, called the *elliptic function*, denoted as *sn*. In that case, in place of Eq. (A.5) we have

$$sn(u, k) = \sin \theta = x \tag{A.6}$$

There are other elliptic functions; for example, the one denoted as *cn* which is defined by

$$cn(u, k) = \cos \theta \tag{A.7}$$

Elliptic integral of the **second kind** is

$$E(x, k) = \int_0^x \sqrt{\frac{1 - k^2 x'^2}{1 - x'^2}} dx' \quad k^2 < 1 \tag{A.8}$$

or its equivalent

$$E(\theta, k) = \int_0^\theta \sqrt{1 - k^2 \sin^2 \theta'}\, d\theta' \tag{A.9}$$

where $x = \sin \theta$. The elliptic integral of the **third kind** is

$$\prod(x, n, k) = \int_0^x \frac{dx'}{(1 + nx'^2)\sqrt{(1 - x'^2)(1 - k^2 x'^2)}} \tag{A.10}$$

Its equivalent is

$$\prod(\theta, n, k) = \int_0^\theta \frac{d\theta'}{(1 + n\sin^2 \theta')\sqrt{(1 - k^2 \sin^2 \theta')}} \tag{A.11}$$

Again, $x = \sin\theta$. Tabulated values of elliptic integrals are available in mathematical handbooks.

Appendix B

Perturbation Theory

The majority of systems in classical mechanics, as already indicated, cannot be solved exactly. Perturbation procedure is an approximation method for obtaining solutions of such systems.

Principle Often it is possible to represent a given Hamiltonian H in the form of an integrable unperturbed part H_0 plus a small non-integrable perturbation H_1 :

$$H(p, q) = H_0(p, q) + \varepsilon H_1(p, q) \qquad \text{(B.1)}$$

where *e* is the perturbation parameter and is assumed to be <<1. For example, the motion of the earth about the sun is an exactly integrable two-body problem. However, in the case of Jupiter, the influence due to other planets and 16 moons is not negligible and can be considered a small perturbation on the two-body problem. The perturbation theory considers techniques for obtaining approximate solutions to H in the form of exact solutions to H_0 plus some corrections due to $H_1(p, q)$. In other words, in the procedure the integrable system plays an important role in solving the non-integrable system.

The basic idea of perturbation theory is to expand the solution $x(t)$ in a power series in *e:* $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$ (B.2)

where $x_0(t)$ is the exact solution to the integrable part H_0 and the corrections $x_1(t)$, $x_2(t)$,... are calculated by a recursion procedure. If *e* is very small, only a few terms in the expansion in Eq. (B.2) will contribute to the value of $x(t)$ and in the limit *e Æ* 0 only the integrable part of the problem remains. It may be noted here that the series may not converge always even for a very small value of *e*. In Eq. (B.2), the first, second, third and subsequent terms are respectively called the zeroth order, first order, second order and higher order corrections to the problem. To illustrate the procedure we shall apply it to some simple cases.

Regular Perturbation Series–An Example As an example of the perturbation procedure, consider the quadratic equation

$$x^2 + x - 6\varepsilon = 0 \qquad \text{(B.3)}$$

where $\varepsilon << 1$. The part

$$x^2 + x = 0$$

can be taken as the integrable zeroth order problem since it gives $x = 0$ and -1 as the two roots. Next, we shall consider a power series expansion of the form

$$x = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots = \sum_{n=0}^{\infty} a_n\varepsilon^n \qquad \text{(B.4)}$$

as the solution of the perturbed problem in Eq. (B.3), where a_0 is the zeroth order root about which we are expanding. Substituting Eq. (B.4) in Eq. (B.3), we have

$$(a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots)^2 + (a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots) - 6\varepsilon = 0 \qquad \text{(B.5)}$$

Equating each power of ε to zero we get

$$\text{coefficient of } \varepsilon^0 : a_0^2 + a_0 = 0 \qquad \text{(B.6a)}$$

$$\text{coefficient of } \varepsilon^1 : 2a_0a_1 + a_1 - 6 = 0 \qquad \text{(B.6b)}$$

$$\text{coefficient of } \varepsilon^2 : a_1^2 + 2a_0a_2 + a_2 = 0 \qquad \text{(B.6c)}$$

Equation (B.6a) gives $a_0 = 0$ and -1, the expected zero order roots. From Eqs. (B.6b) and (B.6c)

$$\text{for } a_0 = 0:\ a_1 = 6,\ a_2 = -36$$

$$\text{for } a_0 = -1:\ a_1 = -6,\ a_2 = 36$$

Corresponding to these two sets, we have the two solutions

$$x_1 = 6\varepsilon - 36\varepsilon^2 + \ldots \qquad \text{(B.7)}$$

$$x_2 = -1 - 6\varepsilon + 36\varepsilon^2 + \ldots \qquad \text{(B.8)}$$

In the limit $\varepsilon \to 0$, these solutions tend to the zeroth order values, as required.

Regular Perturbation Series for Differential Equation Consider the first order differential equation

$$\frac{dx}{dt} = x + \varepsilon x^2 \qquad 0 < \varepsilon << 1 \qquad \text{(B.9)}$$

with the initial condition $x(0) = a$. Expanding the solution $x(t)$ in a power series of the type in Eq. (B.2)

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots \tag{B.10}$$

Substituting this expansion in Eq. (B.9)

$$\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \dots = (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) + \varepsilon (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2$$

Equating the powers of ε, we get

$$\text{Power of } \varepsilon^0 : \dot{x}_0 = x_0 \tag{B.11a}$$

$$\text{Power of } \varepsilon^1 : \dot{x}_1 = x_1 + x_0^2 \tag{B.11b}$$

$$\text{Power of } \varepsilon^2 : \dot{x}_2 = x_2 + 2x_1 x_0 \tag{B.11c}$$

The zeroth order part, Eq. (B.11a), gives

$$\frac{dx_0}{x_0} = dt$$

Integrating and using the initial condition

$$x_0(t) = ae^t \tag{B.12}$$

Substitution of this value of $x_0(t)$ in Eq. (B.11b)

$$\frac{dx_1}{dt} = x_1 + a^2 e^{2t} \tag{B.13}$$

From Eq. (B.10), we have

$$x(0) = x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \dots \tag{B.14}$$

Since $x_0(0) = a$, the initial condition $x_0(0) = a$, will be satisfied by all values of e only if $x_n(0) = 0$ for $n \geq 1$. With the initial condition $x_1(0) = 0$, the solution of the inhomogeneous Eq. (B.13) gives

$$x_1(t) = a^2 e^t (e^t - 1) \tag{B.15}$$

Substitution of Eqs. (B.12) and (B.15) in Eq. (B.11c) gives

$$\frac{dx_2}{dt} = x_2 + 2a^3 e^{2t} (e^t - 1) \tag{B.16}$$

Equation (B.16) when solved gives

$$x_2(t) = a^3 e^t (e^t - 1)^2 \tag{B.17}$$

Combining Eqs. (B.10), (B.12), (B.15) and (B.17), we get

$$x(t) = ae^t [1 + \varepsilon a(e^t - 1) + \varepsilon^2 a^2 (e^t - 1)^2] + \dots \tag{B.18}$$

which is the solution of Eq. (B.9).

which is the solution of Eq. (B.9).

Perturbed Harmonic Oscillator The differential equation representing a harmonic oscillator is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{B.19}$$

Let a cubic perturbation act on the system. On including the perturbation, the differential equation takes the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x - \varepsilon x^3 = 0 \tag{B.20}$$

Again, assuming a solution of the type given in Eq. (B.14) and substituting it in Eq. (B.20), we have

$$\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \omega_0^2\,(x_0 + \varepsilon x_1 + \varepsilon x_2) - \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^3 = 0 \tag{B.21}$$

Equating the coefficients on both sides

coefficient of ε^0 gives: $$\ddot{x}_0 + \omega_0^2 x_0 = 0 \tag{B.22a}$$

coefficient of ε^1 gives: $$\ddot{x}_1 + \omega_0^2 x_1 = x_0^3 \tag{B.22b}$$

coefficient of ε^2 gives: $$\ddot{x}_2 + \omega_0^2 x_2 = 3x_0^2 x_1 \tag{B.22c}$$

The zeroth order equation denoted by Eq. (13.22a) has the solution

$$x_0(t) = a\cos\omega_0 t \tag{B.23}$$

for the initial conditions $x_0(0) = a$ and $\dot{x}_0 = 0$. Substitution of Eq. (B.23) in Eq. (B.22b) gives

$$\ddot{x}_1 + \omega_0^2 x_1 = a^3\cos^3\omega_0 t \tag{B.24}$$

Since

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta$$

equation (B.24) reduces to

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{a^3}{4}(\cos 3\omega_0 t + 3\cos\omega_0 t) \tag{B.25}$$

Equation (B.25) is a linear inhomogeneous equation whose solution is of the form

$$x_1(t) = a_1\cos\omega_0 t + \sin\omega_0 t - \frac{a^3}{32\omega_0^2}\cos 3\omega_0 t + \frac{3a^3}{8\omega_0}\omega_0 t\sin\omega_0 t \tag{B.26}$$

The non-periodic term $t \sin w_0 t$ in the solution is because of the term $3 \cos w_0 t$ in Eq. (B.25), which is in resonance with the intrinsic oscillator frequency. This can be avoided by expanding both the amplitude x and frequency w, which leads to a well-behaved periodic solution for Eq. (B.22b). This procedure can be continued to higher orders in ε with the necessary corrections for eliminating non-periodic terms. The procedure is quite cumbersome.

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Answers to Problems

CHAPTER 1

1. $V = -\left(\frac{ax^2}{2} + \frac{by^2}{2} + \frac{cz^2}{2}\right) + c \qquad c = \text{constant}$

2. $V = -\left(\frac{ax^2}{2} + ayz + bxy^2 + \frac{bz^3}{3}\right) + c \qquad c = \text{constant}$

4. $x(t) = \frac{F_0 t}{m\lambda} + \frac{F_0}{m\lambda^2}\left(e^{-\lambda t} - 1\right) \qquad \mathbf{v}(t) = \frac{F_0}{m\lambda}\left(1 - e^{-\lambda t}\right)$

5. $\mathbf{v}(t) = \frac{g}{k}\left(1 - e^{-kt}\right), \quad x(t) = \frac{gt}{k} - \frac{g}{k^2}\left(1 - e^{-kt}\right) \qquad k = \text{constant}$

6. $\mathbf{v}(t) = \mathbf{v}_0 e^{-(k/m)t} \qquad x(t) = x_0 + \frac{m\mathrm{v}_0}{k}\left[1 - e^{-(k/m)t}\right] \qquad k = \text{constant}$

7. $\mathbf{v}(t) = \mathbf{v}_0 - \frac{\mathbf{v}_0^{1/2} kt}{m} + \frac{k^2 t^2}{4\ \ m^2} \qquad k = \text{constant}$

   $x(t) = x_0 + \mathbf{v}_0 t - \frac{k\mathbf{v}_0^{1/2} t^2}{2m} + \frac{k^2 t^3}{12m^2}$

8. $\mathbf{v}(t) = \frac{A}{m\omega}(1 - \cos\omega t), \quad x(t) = x_0 + \frac{At}{m\omega} - \frac{A}{m\omega^2}\sin\omega t$

9. $\mathbf{v}(t) = \frac{bt^2}{2m} \qquad x(t) = \frac{bt^3}{6m}$

10. $\frac{a}{2} + b$

11. (i) $\frac{2\pi}{\sqrt{3ka}}$ (ii) $ma^2\sqrt{3ka}$

13. $\frac{f^2 t_0^3}{3m}$

CHAPTER 2

1. $6\times10^5\,\text{N}$ $13.54\times10^3\,\text{m/s}$

2. (i) 2.72 (ii) 7.4

5. $\mathbf{a}=\frac{m_2 g}{m_1+m_2}$, $T=\frac{m_1 m_2 g}{m_1+m_2}$

6. $\mathbf{R}=\frac{26\hat{i}+16\hat{j}+29\hat{k}}{8}$ $\mathbf{L}=96\hat{i}-9\hat{j}+16\hat{k}$

7. $\mathbf{R}=\frac{7\hat{i}+13\hat{j}+15\hat{k}}{9}$ $\mathbf{V}=\frac{2\hat{i}-18\hat{j}+2\hat{k}}{9}$

   $\mathbf{L}=48\hat{i}+24\hat{j}-50\hat{k}$

8. $10.66\times10^{-22}\,kg.m/s$; 9.1 eV

CHAPTER 3

1. $\delta x = \cos\theta\,\delta r - r\sin\theta\,\delta\theta \qquad \delta y = \sin\theta\,\delta r + r\cos\theta\,\delta\theta$

2. $\delta\rho = \frac{x}{\sqrt{x^2 + y^2}}\,\delta x + \frac{y}{\sqrt{x^2 + y^2}}\,\delta y$

$$\delta\phi = -\frac{y}{x^2 + y^2}\,\delta x + \frac{x}{x^2 + y^2}\,\delta y, \quad \delta z = \delta z$$

3. $\frac{m_1}{m_2} = \frac{\sin\beta}{\sin\alpha} \quad \alpha, \beta =$ Angles of the inclines

4. $T = (1/2) Mg \cot\alpha$

5. $Q_r = F_x \cos\theta + F_y \sin\theta = F_r$

$$Q_\theta = -rF_x \sin\theta + rF_y \cos\theta = rF_\theta$$

6. $Q_r = F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z \cos\theta$

$$Q_\theta = F_x\, r\cos\theta\,\cos\phi + F_y r\cos\theta\,\sin\phi - F_z r\sin\theta$$

$$Q_\phi = -\,F_x r\sin\theta\,\sin\phi + F_y r\sin\theta\,\cos\phi$$

7. $m\ddot{r} - mr\dot{\theta}^2 + \dfrac{k}{r^2} = 0 \qquad mr^2\ddot{\theta} + 2mr\,\dot{r}\dot{\theta} = 0$

8. (i) $m\ddot{x} + kx = 0$ (ii) $m\ddot{x} + kx = A_0 \sin\omega t$

9. $L = \dfrac{1}{2}(m_1 + m_2)\,\dot{x}^2 + m_1 g x + m_2 g\,(l - x)$,

$$(m_1 + m_2)\ddot{x} - (m_1 - m_2)\,g = 0 \qquad \ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

10. $L = \dfrac{1}{2} m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - V(\rho, \phi, z)$

$$\frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\phi}^2 + \frac{\partial V}{\partial \rho} = 0 \qquad \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) + \frac{\partial V}{\partial \phi} = 0 \qquad \frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} = 0$$

11. $L = \dfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta\right) - \dfrac{1}{2} kr^2$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\dot{\phi}^2 \sin^2\theta + kr = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mr^2\dot{\phi}^2 \sin\theta\,\cos\theta = 0 \qquad \frac{d}{dt}\left(mr^2\dot{\phi}\,\sin^2\theta\right) = 0$$

12. $R = \dfrac{\mathbf{v}_0^2 \sin 2\alpha}{g} \qquad T = \dfrac{2\mathbf{v}_0 \sin\alpha}{g}$

13. $\ddot{x} = \dfrac{2}{3} g\sin\theta, \quad \ddot{\phi} = \dfrac{2g\sin\theta}{3a}, \quad \mathbf{v} = \left(\dfrac{4gl\sin\theta}{3}\right)^{1/2} \quad l = \text{length}$

14. $L = \dfrac{1}{2} m_1 l_1^2\, \dot{\theta}^2 + \dfrac{1}{2} m_2\left[l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1 l_2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right]$

$$-\, m_1 g\,(l_1 + l_2 - l_1\cos\theta) - m_2 g\,(l_1 + l_2 - l_1\cos\theta - l_2\cos\phi)$$

$$(m_1 + m_2) l_1^2\ddot{\theta} + m_2 l_1 l_2\,\ddot{\phi}\cos(\theta - \phi) + m_2 l_1 l_2\dot{\phi}^2 \sin(\theta - \phi)$$

$$+\,(m_1 + m_2)\,g l_1 \sin\theta = 0$$

$$l_2\ddot{\phi} + l_1\,\ddot{\theta}\cos(\theta - \phi) - l_1\dot{\theta}^2\,\sin(\theta - \phi) + g\sin\phi = 0$$

where θ and ϕ are angles with the vertical.

15. $m\ddot{x} = -\dfrac{\partial V}{\partial x}$

16. $\ddot{\phi} = -\frac{2g\phi}{3(R-a)}$ $\omega = \sqrt{\frac{2g}{3(R-a)}}$

17. (i) $L = ma^2\dot{\theta}^2 (1-\cos\theta) - mga(1+\cos\theta)$

(ii) $\ddot{\theta}(1-\cos\theta) + \frac{1}{2}\dot{\theta}^2 \sin\theta - \frac{g}{2a}\sin\theta = 0$

18. $M\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0$ $m\ddot{x}_2 + kx_2 - kx_1 = 0$

CHAPTER 4

1. $m\ddot{x} + kx = 0$

2. $x = a\cosh\left(\frac{y-b}{a}\right)$

3. $m\ddot{x} - mg\sin\alpha + \lambda = 0,$ $mb^2\ddot{\theta} - \lambda r = 0;$ $\mathbf{v} = \sqrt{gl\sin\alpha}$

4. $\lambda = \frac{mg\sin\alpha}{3}$ $\mathbf{v} = \left(\frac{4gl\sin\alpha}{3}\right)^{1/2}$

6. (i) $\lambda_1 = mg(2\cos\theta - 1)$ (ii) $\theta = 60°$

7. $n_i = \frac{g_i}{\exp(\lambda_1 + \lambda_2 \in_i) + 1}$ λ_1 and λ_2 are Lagrange multipliers

8. $\lambda = mg(3\cos\theta - 1)$ $h = \frac{4}{3}a$

CHAPTER 5

1. $F(r) = \frac{-\mathbf{L}^2}{m}\left(\frac{2k^2}{r^5} + \frac{1}{r^3}\right)$ $\mathbf{L}$ = angular momentum

2. $F(r) = \frac{-\mathbf{L}^2}{mr^3}\left(\alpha^2 + 1\right)$ $\mathbf{L}$ = angular momentum

5. (i) $\in = 0.1$ (ii) $a = 9596$ km (iii) $b = 9548$ km

   (iv) $T = 2\text{h}\ 36\text{m}\ 6\text{s}$

9. $a = \left(b^2 + \frac{k}{m\mathbf{v}_0^2}\right)^{1/2}$

12. $E = 0$

13. $F(r) = \frac{-L^2}{lm}\frac{1}{r^2}$

15. (i) $\in = 0.0635$; (ii) $a = 7880$ km, $b = 7848$ km ; (iii) $T = 1\text{h}\ 56\text{m}\ 8\text{s}$

17. (i) $\in = 0.473$; (ii) 19610.2 km; (iii) 3271.5 m/s

CHAPTER 6

1. $H(x, p_x) = \frac{p_x^2}{2m} + \frac{1}{2}kx^2; \qquad m\ddot{x} + kx = 0$

2. Concentric ellipses

3. $H = \frac{p_\theta^2}{2ma^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(z^2 + a^2)$

$$\dot{\theta} = \frac{p_\theta}{ma^2}, \qquad \dot{p}_\theta = 0; \qquad \dot{z} = \frac{p_z}{m}, \qquad \dot{p}_z = -kz$$

4. $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) - \frac{qB}{2m}L_0, \quad L_0$ – angular momentum

5. (i) $p_i = P_i, Q_i = q_i$ (ii) $p_i = -P_i, q_i = -Q_i$

6. Yes, $F = q/Q$

7. $F_3 = -(e^Q - 1)^2 \tan p + \text{constant}$

9. $[Q, P] = 1$

11. (i) $[L_x, p_y] = p_z$ (ii) $[x, L_y] = z$

13. $F_2 = -\cos Pe^{-q}$

15. $ad - bc = 1$

16. $\dot{r} = \frac{p_r}{m} \qquad \dot{p}_r = \frac{p_\theta^2}{mr^3} + mg\cos\theta - k(r - r_0) \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$

$$\dot{p}_\theta = -mgr\sin\theta \qquad m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r - r_0) = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0$$

17. $ml^2\ddot{\theta} = ml^2\dot{\phi}^2 \sin\theta\cos\theta - mgl\cos\theta$

$p_\phi^2 = \text{constant}$ (for θ, refer to Example 3.10)

CHAPTER 7

2. $$\frac{1}{2m}\left[\left(\frac{dW_1}{dr}\right)^2+\frac{1}{r^2}\left(\frac{dW_2}{d\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{dW_3}{d\phi}\right)^2\right]+V(r)=\alpha_1=E$$

$$W_3=\alpha_\phi\phi; \qquad W_2=\int\sqrt{\alpha_\theta^2-\frac{\alpha_\phi^2}{\sin^2\theta}}\,d\theta \qquad \alpha_\theta,\alpha_\phi=\text{constants}$$

$$W_1=\int\sqrt{2m(E-V)-\frac{\alpha_\theta^2}{r^2}}\,dr$$

3. $$y=\mathbf{v}_0\,(\beta_1+t)\sin\alpha-\frac{1}{2}g(\beta_1+t)^2 \qquad \beta_1=\text{constant}$$

4. $$W=W_1(\theta)+W_\phi(\phi)+W_\psi(\psi) \qquad t+\beta_1=\frac{\partial W}{\partial\alpha_1}=\int_{\theta_0}^{\theta}\frac{I\,d\theta}{f(\theta)}$$

$$f(\theta)=\left[2I\alpha_1-\frac{(\alpha_\phi-\alpha_\psi\cos\theta)^2}{\sin^2\theta}-\frac{I}{I_3}\alpha_\psi^2-2IMgl\cos\theta\right]^{1/2}$$

$$\beta_2=\frac{\partial W}{\partial\alpha_\phi}=-\int_{\theta_0}^{\theta}\frac{(\alpha_\phi-\alpha_\psi\cos\theta)\,d\theta}{f(\theta)\sin^2\theta}+\phi$$

$$\beta_3=\frac{\partial W}{\partial\alpha_\psi}=\int_{\theta}^{\theta}\frac{(\alpha_\phi-\alpha_\psi\cos\theta)\cos\theta\,d\theta}{f(\theta)\sin^2\theta}-\frac{I}{I_3}\alpha_\psi\int_{\theta_0}^{\theta}\frac{d\theta}{f(\theta)}+\psi$$

$$I_1=I_2=I\neq I_3 \qquad \theta,\phi,\psi=\text{Eulers angles}$$

CHAPTER 8

1. $\omega = \sqrt{\dfrac{Mgl}{I}}, \quad \dfrac{I}{Ml}$

2. $I = \begin{pmatrix} 16 & -9 & 1 \\ -9 & 17 & 1 \\ 1 & 1 & 19 \end{pmatrix}$

3. $I_{11} = \dfrac{M}{3}(b^2 + c^2) \qquad I_{22} = \dfrac{M}{3}(a^2 + c^2) \qquad I_{33} = \dfrac{M}{3}(a^2 + b^2)$

$I_{12} = I_{21} = -\dfrac{Mab}{4} \qquad I_{13} = I_{31} = -\dfrac{Mac}{4} \qquad I_{23} = I_{32} = -\dfrac{Mbc}{4}$

4. (i) $\mathbf{L} = \omega I_{xx}\hat{i} + \omega I_{xy}\hat{j} + \omega I_{zx}\hat{k}$

(ii) $I_{xy} = I_{zx} = 0$ (iii) $T = \dfrac{1}{2} I_{xx}\omega^2$

5. $I_{11} = \dfrac{M}{12}(b^2 + c^2) \qquad I_{22} = \dfrac{M}{12}(a^2 + c^2) \qquad I_{33} = \dfrac{M}{12}(a^2 + b^2)$

$I_{12} = I_{23} = I_{13} = I_{21} = I_{32} = I_{31} = 0$

6. $T = \dfrac{1}{2} I(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \dfrac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$

CHAPTER 9

1. Translational moton $\omega_1 = 0$; Vibrational motion $\omega_2 = \sqrt{k/\mu}$.

3. $\omega_1^2 = \frac{k}{2m}(3+\sqrt{5}) \qquad \omega_2^2 = \frac{k}{2m}(3-\sqrt{5})$

$\frac{1-\sqrt{5}}{2} \qquad \frac{1+\sqrt{5}}{2}$

4. $$\omega^2 = \frac{g(m_1+m_2)(l_1+l_2)}{2m_1 l_1 l_2} \pm \frac{\left[g^2(m_1+m_2)^2\,(l_1+l_2)^2 - 4m_1\,(m_1+m_2)\,l_1 l_2 g^2\right]^{1/2}}{2m_1 l_1 l_2}$$

(i) $\omega_1 = \sqrt{g/l_2} \qquad \omega_2 = \sqrt{g/l_1}$

(ii) $\omega_1^2 \cong \frac{gm_2}{m_1}\left(\frac{1}{l_1}+\frac{1}{l_2}\right) \qquad \omega_2^2 = \frac{g}{l_1+l_2}$

(iii) $\omega_1^2 = \frac{g}{l}(2+\sqrt{2}) \qquad \omega_2^2 = \frac{g}{l}(2-\sqrt{2})$

5. $\omega_1 = \sqrt{g/l} \qquad \omega_2 = \sqrt{\frac{g}{l}+\frac{k}{\mu}} \qquad \mu = \frac{m_1 m_2}{m_1+m_2}$

6. $\omega_1 = \sqrt{\frac{2k}{m}} \qquad \omega_2 = \sqrt{\frac{k}{m}(2+\sqrt{2})} \qquad \omega_3 = \sqrt{\frac{k}{m}(2-\sqrt{2})}$

CHAPTER 10

1. $u = 0.846\ c$
2. $L_0 = 120.8$ m
3. $\mathbf{v} = 0.2\ c$
4. $u = 0.8\ c$
5. $u' = 0.8\ c$
6. $\mathbf{v} = 0.87\ c$
7. $p = 1626.4\ \frac{\text{MeV}}{c}$
8. $\Delta t = 1.67 \times 10^{-6}\text{s}$
9. $\mathbf{v} = 0.5\ c$
10. 0.512 MeV; 939.4 Mev
11. T = 901.1 MeV
12. $\mathbf{v} = 0.996\ c$
13. $T = 1.44 \times 10^{-11}$ J
14. $\mathbf{v} = 0.866\ c$
15. $\mathbf{v} = 0.866$ c
16. $m = 2m_0$; $\mathbf{v} = 0.866\ c$
17. $\mathbf{v} = 0.866\ c$; p = 4.734×10^{-22} kms^{-1}
18. $\Delta t = 3 \times 10^{-6}\text{s}$
19. 54.62 m
20. $\mathbf{v} = 0.141\ c$
21. $\Delta \tau = 2.18 \times 10^{-6}\text{s}$
22. $\mathbf{v} = 0.553\ c$
23. $\rho = \frac{\rho_0}{1 - \mathbf{v}^2/c^2}$
24. $\Delta m = 4.44 \times 10^9\text{kg}$
25. $p = 10.5$ MeV/c $u = 0.9988\ c$
27. $m = 1954\ m_e$ $u = 0.9999998$ c
29. m = 4.42×10^{-36}kg 2.208×10^{-32} kg

32. $T_\mu = 4.084$ MeV $\qquad E_\nu = 29.70\ MeV$

33. $u = 0.92\ c$

34. 199947 m

CHAPTER 11

3. (0, 0) – stable spiral if $\beta > \alpha^2/4$

– stable node if $0 < \beta < \alpha^2/4$

– saddle point if $\beta < 0$

$(\pm\sqrt{-\beta/\gamma}, 0)$ – stable spiral $\beta < 0, 2|\beta| > \alpha^2/4$

– stable node $\beta < 0, 2|\beta| < \alpha^2/4$

– saddle point $\beta > 0$

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